Incomplete markets, liquidation risk, and the term structure of interest rates

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Abstract

We analyse the term structure of interest rates in a general equilibrium model with incomplete markets, borrowing constraint, and positive net supply of government bonds. Uninsured idiosyncratic shocks generate bond trades, while aggregate shocks cause fluctuations in the trading price of bonds. Long bonds command a "liquidation risk premium" over short bonds, because they may have to be liquidated before maturity –following a bad idiosyncratic shock– precisely when their resale value is low –due to the simultaneous occurrence of a bad aggregate shock. Our framework endogenously generates limited cross-sectional wealth heterogeneity among the agents (despite the presence of uninsured idiosyncratic shocks), which allows us to characterise analytically the shape of the entire yield curve, including the yields on bonds of arbitrarily long maturities. Agents' desire to hedge the idiosyncratic risk together with their fear of having to liquidate long bonds at unfavourable terms imply that a greater bond supply raises the level of the yield curve, while an increase in the relative supply of long bonds raises its slope.

Keywords: Incomplete markets; Borrowing constraint; Yield curve.
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1 Introduction

This paper analyses the term structure of real interest rates in an infinite-horizon, general equilibrium framework in which agents are hit by aggregate shocks, as well as idiosyncratic shocks that cannot be fully insured due to incomplete markets and borrowing constraint. On the one hand, uninsured idiosyncratic shocks generate bond trades –as traders willing to buy bonds for precautionary purposes purchase them from traders willing to sell to buffer the shocks. On the other hand, aggregate shocks cause fluctuations in bond prices, and hence induce some volatility in the terms at which bond trades take place.

The key novelty of our approach is the construction of a tractable equilibrium allowing for an analytical characterisation of the entire yield curve (from one-period to arbitrarily long bonds), while accommodating active trades of positive net bond supplies at all maturities. Tractability follows from two main underlying assumptions, which jointly ensure that the model generates a finite-dimensional cross-sectional distribution of wealth as an equilibrium outcome.[1] The first assumption is that agents' instant utility is separable in consumption and labour and linear in labour (as in, e.g., Scheinkmann and Weiss (1986)). As we show, endogenous labour supply with quasi-linear preferences implies that agents entering the good idiosyncratic state ("employment") are willing to work as much as necessary to instantaneously replete their bond portfolio; in consequence, bond holdings are homogeneous across agents in that state –and independent of their *history* of idiosyncratic states. The second assumption is that equilibrium bond holdings are sufficiently small for agents entering the bad idiosyncratic state ("unemployment") to be willing to liquidate their asset wealth in a small number of periods. The reason is that, under transitory idiosyncratic shocks, agents in the bad idiosyncratic state may be willing to *borrow* against future income, in which case they hit the borrowing constraint. Agents in this situation simply liquidate their bond portfolio and, by way of consequence, no longer affect bond prices. We focus on the equilibria with "full asset liquidation", where small equilibrium bond holdings in the first place ensures that agents immediately face a binding borrowing constraint when hit by a bad idiosyncratic shock.[2]

The theoretical investigation of the model shows that, in this framework i) a higher net supply of government bonds of any maturity –financed by lump-sum taxes– raises the level of the entire yield curve; ii) a higher net supply of long bonds raises the slope of the yield curve –where we define the "slope" as the yield difference between the two ends of the curve.[3] The first effect (on the *level* of the curve) can be seen as a generalisation to the full set of bond maturities of a

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[1] This is in contrast with most incomplete market economies, where any agent's wealth depends on his entire own history of idiosyncratic shocks, so that infinitely many agent types ultimately coexist in the economy. Such models must be solved numerically and can accommodate only a small number of assets, typically one or two (e.g., Den Haan (1996), Heaton and Lucas (1996), Krusell and Smith (1997), Heathcote (2005)).

[2] In the separate technical appendix to the paper, we study numerically a relaxed model wherein asset liquidation is gradual rather than immediate, and confirm all the results obtained in the case of instantaneous asset liquidation.

[3] Empirically, Laubach (2009) reports that higher levels of public debt or larger fiscal deficits significantly raise real interest rates. Relatedly, Krishnamurthy and Vissing-Jorgensen (2012) show that the size of public debt negatively affects the spread between corporate and Treasury bond yields, a reflection of the presence of a "Treasury demand function". The work of Engen and Hubbard (2005), Gale and Orszag (2003) and Longstaff (2004) point toward a similar relationship.

common implication of non-Ricardian economies, according to which an increase in the supply of assets available for self-insurance lowers the price of such assets and raises their expected return.[4]

The identification of the second effect (on the *slope* of the yield curve) is our central substantive result and can be explained as follows. Under aggregate and uninsured idiosyncratic shocks, holders of long bonds are exposed to a specific source of risk, namely, that of having to sell bonds –due to the occurrence of a bad idiosyncratic shocks– precisely at a time when the resale value of bonds is low –because the aggregate state is itself unfavourable. Decreasing marginal utility implies that the utility loss incurred when selling at a low price is not compensated by the potential benefit from selling bonds at high prices. More formally, the source of risk that we identify results from the covariance between the resale price of long bonds and bondholders' wealth. Under incomplete insurance and active trading, bondholders' wealth explicitly enters future marginal utility, because it determines what consumption can be achieved, should part or all of their bond wealth be liquidated (after a bad idiosyncratic shock). In this case, the combination of aggregate and uninsured idiosyncratic shocks generates negative co-movements between bondholders' pricing kernel and the trading price of long bonds, which consequently depresses their average prices –i.e., it raises their average yields. These co-movements being driven by the trading price of bonds, they are present even if the bonds' own *income* is riskless, as we assume it to be (by considering zero-coupon bonds paying a certain terminal payoff). In contrast to long bonds, one-period bonds are never resold, implying no such co-movements and no such risk. We refer to the risk of having to liquidate long bonds at uncertain future resale prices as *liquidation risk*, and to the associated yield premium as the *liquidation risk premium*. By linking the yield premium on long bonds to the degree of certainty at which bonds of different maturities can be converted into consumption, our model provides a possible foundation for the notion that investors prefer short to long bonds because the former are more "liquid" than the latter.

Under which conditions do long yields incorporate a liquidation risk premium, and how does the latter vary with the net supply of bonds? Trivially, there is no liquidation risk without aggregate shocks, since in this case all bond prices are constant –and hence do not covary with bondholders' wealth. Similarly, there is no such risk under complete markets, because full risk sharing implies that agents never have to sell bonds after a bad idiosyncratic shock. More subtly, when markets are incomplete but the equilibrium features no trade –a popular specification–[5], bond prices and yields typically differ from their complete-market counterparts, but again agents never actually rebalance their portfolio (by the no-trade property) and hence no liquidation risk is present. It follows that *aggregate risk*, *uninsured idiosyncratic risk* and *positive net supplies of long bonds* are all needed for the liquidation risk premium to kick in. Under these conditions, the greater the supply of long bonds, the greater the (negative) covariance between bondholders' pricing kernel and the resale price of long bonds, and the greater the liquidation risk premium that these bonds command.

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[4] For example, the level effect on interest rates is a standard property of overlapping-models, which we discuss further in the literature review below.

[5] Constantinides and Duffie (1996), and more recently Krusell et al. (2011) analyse equilibrium asset prices in the no-trade equilibria of incomplete-market economies. We return to these contributions in the literature review below.

In order to best disentangle the respective roles of aggregate risk, idiosyncratic risk and bond supplies in affecting the shape of the yield curve, we proceed gradually as follows. We first characterise the yield curve in a complete-market version of our model, where the demand for bonds is only driven by the representative agent's desire to hedge the *aggregate risk*. We notably verify that the yield curve is "well behaved" in this scenario and, in particular, that it is consistent with Ricardian equivalence in that the yield curve is independent of the net supply of bonds. Starting from this benchmark, we study how the shape of the yield curve changes when insurance markets against the idiosyncratic risk are shut down, while bonds of all maturities are kept in zero net supply. Because private agents cannot have negative asset wealth (by assumption), the outcome is a no-trade equilibrium wherein agents' demands for bonds are driven by their desire to hedge the aggregate risk *and the idiosyncratic risk* (with bond prices adjusting up to the point where net demands are all equal to zero). This environment shows that the two hedging motives for holding bonds interact in a nontrivial way, and typically affect the two ends of the yield curve differently.[6] Finally, starting from this incomplete-market, zero-net-supply case, we analyse how a gradual *increase in bond supplies* alters the shape of the yield curve, leading to the level and slope effects discussed above. For each of these environments (complete markets, incomplete markets/zero net supply, incomplete markets/positive net supply), we show how changes in the deep parameters of the model affect the level and the volatility of the pricing kernel that determines all bond prices. In particular, for the two cases where markets are incomplete, we structure our discussion around a factorisation of the pricing kernel in the spirit of Constantinides and Duffie (1996) and Krueger and Lustig (2010), which allows us to disentangle how the two hedging motives interact and ultimately affect the shape of the curve. In all three specifications, we also derive explicit formulas for the long and the average short yield in the case of small, i.i.d. aggregate shocks, where the various effects at work on those yields are perfectly transparent.

In what follows, we first discuss the related literature (Section 2), then introduce our model (Section 3), and finally analyse the shape of the yield curve under the three configurations just discussed (Section 4). We conclude the paper with a brief summary of our results (Section 5).

2   Related literature

Because of their inherently non-Ricardian nature, overlapping generations (OLG) models have frequently been used to study the effect of fiscal policy on the real yield curve. For example, even simple OLG models with two or three-period lived agents typically have the property that increasing the stock of government bonds can raise the equilibrium interest rate when agents are constrained by the supply of stores of value in the economy (Barro (1974)). Our framework and results differ from such simple OLG models in the following respects. First, these models are useful for analysing the long-run, demographic determinants of the yield curve (see, e.g., Guibaud,

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[6]In particular, we show that time-variations in idiosyncratic risk can in itself raise the term premium (even under zero net bond supply), in the same vein as Mankiw (1986) showed that it could raise the equity premium.

Nosbusch and Vayanos (2013)); however, they are arguably ill-suited for capturing the bond price risk that occurs at the much shorter business cycle frequency, and on which our analysis focuses. Second, there is no continuity between such simple OLG models and the frictionless, Ricardian benchmark, so that the degree of departure from the latter environment cannot be perfectly controlled; in contrast, our framework nests both the Ricardian model and a version of the simple OLG model as special cases.[7] On the other hand, "perpetual youth" model of the kind studied by Blanchard (1985), Weil (1989) and more recently Gârleanu and Panageas (2012), retain the tractability of simple OLG models and do nest the Ricardian benchmark (when the birth and death rates are both set to zero). However, there is no liquidation risk in these models, because agents never have to liquidate their portfolio to finance current consumption (instead wealth is either kept for ever as in Weil, or seized at death and redistributed to the living according to an actuarially fair life insurance scheme as in Blanchard). Finally, while finite-life, multi-period OLG models have realistic time scales and can in principle allow for random liquidation before death, they typically cannot be solved in closed form and thus the number of assets under scrutiny must remain small (e.g., Storesletten, Telmer, and Yaron (2007), Gomes and Michaelides (2008)).

The idea that uninsured idiosyncratic risk can help explain asset-pricing puzzles was first explored in finite-horizon economies. Following the lead of Mankiw (1986) and Weil (1992), who focused on stock returns, Heaton and Lucas (1992), and more recently Holmström and Tirole (2001), have used three-period models to analyse the effects of the interactions between idiosyncratic and aggregate risks on the yield curve. These models provide important insights into these interactions, and usually allow for positive asset supplies and active trades among heterogeneous agents; however, they leave open the question of how they affect the yield curve over a long horizon.

There is a key class of infinite-horizon, incomplete-market models where analytical expressions for the price of long assets can be obtained: those where the no-trade equilibrium prevails. Such is the case in Constantinides and Duffie's (1996) model of the equity premium, where the no-trade property follows from the fact that idiosyncratic shocks are permanent. A more recent contribution is Krusell, Mukoyama, and Smith (2011), who study asset prices in the autarkic equilibrium of an incomplete-market model with transitory idiosyncratic shocks. In their model agents value assets (including bonds of different maturities) for their ability to transfer wealth across periods and smooth out idiosyncratic income shocks, but do not trade in equilibrium. In contrast, since our focus is on how the quantity of assets available in the market allows this intertemporal smoothing to take place, and thereby affects the desirability and equilibrium price of bonds, our results require active trading of positive net bond quantities following idiosyncratic shocks.

As discussed by Kehoe and Levine (2001), there are two main classes of infinite-horizon economies with limited risk-sharing: "liquidity-constrained" economies which, like ours, feature incomplete markets and, typically, an exogenous debt limit; and "debt-constrained" economies, which have

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[7]To be more specific: our model becomes a frictionless, representative-agent economy when the probability to be hit by a bad idiosyncratic shock is set to zero, and is observationally equivalent to a version of the OLG model with two period lived agents when this probability is set to one.

complete markets but an endogenous debt limit in that agents can only borrow up to the point where they will be willing to repay (rather than revert to autarky). Following Kehoe and Levine (1993) and Alvarez and Jermann (2000), Seppälä (2004) studies the yield curve in a debt-constrained economy where the debt limit varies endogenously over time, and shows numerically that this framework generates time-varying term premia.[8] There are at least two important differences between his approach and ours. First, he does not analyse the impact of the volume and maturity structure of public debt on the shape of the yield curve, which is the main focus of our paper. And second, we combine borrowing constraint and incomplete insurance against idiosyncratic shocks, which leads us to emphasise the liquidity role played by government bonds (a notion that is absent from debt-constrained economies) and the way it generates supply effects on the shape of the yield curve.

In a recent line of research, Vayanos and Vila (2009), and Greenwood and Vayanos (2010) formalise the notion of "preferred habitat" in an environment with limited arbitrage. They use a partial equilibrium model wherein some investors have a preference for specific maturities, and analyse how exogenous variations in the short rate are transmitted to long rates by arbitrageurs. There are at least two important differences between their approach and ours. First, their framework does not allow studying the impact of public debt on the level of the yield curve (as opposed to its slope), since short rates are exogenous; in contrast, we are interested in how public debt affects the whole curve, including its short end. Second, all agents are utility-maximising in our framework, so that investors' preference for some maturities and the sensitivity of the yield curve to the maturity structure of the debt are both endogenously determined by agents' pricing kernel.

In order to organise their empirical findings, Krishnamurthy and Vissing-Jorgensen (2012) develop a theoretical model of liquidity demand that generates a downward-sloping demand for government bonds, as does our model. More specifically, in their model the size of public debt negatively affects the spread between corporate and Treasury bond yields, as is consistent with the "Treasury demand function" that is observed in the data. The key difference between their approach and ours is that their aggregate demand for liquidity is based on the assumption that government bonds directly enter agents' utility, while our model provides a foundation for the liquidity motive for holding bonds based on financial frictions.

Finally, a popular approach in interest rate modelling is to assume the absence of arbitrage and directly considers an exogenous pricing kernel to price bonds of various maturities (see Dai and Singleton (2006) for an overview). Some recent papers following this tradition introduce macroeconomic factors as determinants of the pricing kernel (see Ang and Piazzesi (2003) on monetary policy, and Dai and Philippon (2006) on fiscal policy). In contrast, we build a general equilibrium model with utility-maximizing agents where the pricing kernel is endogenously determined by agents' utility function together with the financial frictions they face.

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[8]In the separate technical appendix to this paper, we show that our framework also generates time-varying risk premia, so that the Expectations Hypothesis does not hold. Krueger and Perri (2011) study fiscal policy in a debt-constrained economy, but do not introduce public debt nor study the shape of the yield curve.

3 The model

We consider a discrete-time economy with a single consumption good and populated by a continuum $I = [0, 1]$ of infinitely-lived agents who face two sources of risk: an aggregate shock that affects the productivity of employed agents as well as the probability to be employed (Section 3.1); and an uninsurable individual shock that causes households to switch idiosyncratically between employment and unemployment (Section 3.2). The government issues and rolls over positive stocks of zero-coupon bonds of different maturities, and adjust taxes so as to maintain a given maturity structure of the debt (Section 3.3). Agents optimally hold and trade these bonds, in part to self-insure against the income variability induced by changes in their employment status (Section 3.4). In equilibrium, the supply of bonds of each maturity must be equal to the economywide demands for each maturity by heterogeneous agents (Section 3.5).

3.1 Aggregate states

The economy is characterized at every date $t = 0, 1, \ldots$ by an aggregate state s_t, where $s_t = h$ if this state is "high" and $s_t = l$ if it is "low". Let $s^t = \{s_0, \ldots, s_t\}$ denote the history of aggregate states from date 0 to date t and S^t the set of all possible histories. The aggregate state evolves according to a first-order Markov chain with transition matrix

$$T = \begin{bmatrix} \pi^h & 1 - \pi^h \\ 1 - \pi^l & \pi^l \end{bmatrix}. \tag{1}$$

We denote by $\eta^h \equiv \left(1 - \pi^l\right) / \left(2 - \pi^l - \pi^h\right)$ and $\eta^l \equiv 1 - \eta^h$ the unconditional fractions of time spent in state h and l, respectively. We also make the following assumption:[9]

Assumption A (Persistence of aggregate state). $\pi^h + \pi^l \geq 1$.

While not necessary for most of our results, Assumption A allows us to focus our analysis on the empirically relevant case where conditional yield curves are monotonic in both states of the world (see Section 4 below). Without this assumption the yield curve may be oscillating in either state or both, a feature that is not observed in the data.[10] We assume that the probability distribution across aggregate states at date 0 is $\left[\eta^l, \eta^h\right]$, and we denote by $\nu_t : S^t \to [0, 1]$ $(t = 0, 1, \ldots)$ the probability measure over aggregate histories up to date t, consistent with the transition matrix T and the initial distribution.

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[9] In what follows, the assumptions we make start holding as soon as they are stated.

[10] This assumption is also supported by direct evidence from estimated empirical Markov-switching business cycle models. For example, Hamilton ((1994), chap. 22) finds $\pi^h + \pi^l = 1.65$ at a quarterly frequency for the US economy.

3.2 Individual states

In every period, each agent can be in either of two states, "employed" or " unemployed". Let e_t^i denote the employment status of agent i at date t, where $e_t^i = 1$ if the agent is employed and $e_t^i = 0$ if the agent is unemployed. Denote by $\alpha_t \left(s^t \right) : S^t \to [0, 1]$, $t = 0, 1, \ldots$ the probability for an agent to be employed at date t when he or she was employed at date $t - 1$ and the history of the aggregate shock is $s^t \in S^t$, i.e., $\alpha_t \left(s^t \right) \equiv \Pr(e_t^i = 1 \big| e_{t-1}^i = 1, s^t)$. Similarly, denote by $\rho_t : S^t \to [0, 1], t = 0, 1, \ldots$ the probability for an agent who was unemployed at date $t - 1$ to stay unemployed at date t, i.e., $\rho_t \left(s^t \right) \equiv \Pr(e_t^i = 0 \big| e_{t-1}^i = 0, s^t)$. The implied transition matrix across employment statuses is

$$\Pi_t \left(s^t \right) = \left[\begin{array}{cc} \alpha_t \left(s^t \right) & 1 - \alpha_t \left(s^t \right) \\ 1 - \rho_t \left(s^t \right) & \rho_t \left(s^t \right) \end{array} \right] .$$

All idiosyncratic changes in employment statuses are independent across agents and the law of large number holds on the continuum of agents. The history of individual shocks up to date t is denoted by $e^{i,t}$, where $e^{i,t} = \{e_0^i, \ldots, e_t^i\} \in \{0, 1\}^t = E^t$. E^t is the set of all possible individual histories up to date t, and $\mu_t^i : E^t \to [0, 1]$ denotes the probability measure of individual histories, consistent with the transition matrix Π_t and an initial probability distribution ω_0. For example, $\mu_t^i \left(e^{i,t} \right)$ is the probability that agent i has experienced the individual history $e^{i,t}$ up to date t.

The individual and aggregate states affect the economy as follows. Employed agents (for whom $e_t^i = 1$) freely choose their labour supply and produce z^{s_t} units of the (single) consumption good per unit of labour supplied in state $s_t = h, l$, with $z^h \geq z^l > 0$.[11][12] Unemployed agents (for whom $e_t^i = 0$) get a fixed "home production" income $\delta > 0$. We now make the following assumptions:

Assumption B (Productivity and idiosyncratic risk). *(i)* $1/z^l < u'(\delta)$; *(ii)* $\alpha^h \geq \alpha^l$ *and* $\rho^h \leq \rho^l$.

Point (i), combined with the fact that $z^h \geq z^l$, will imply that in equilibrium the employed will effectively consume more (and hence enjoy lower marginal utility) than the unemployed in both aggregate states –that is, the equilibrium will feature imperfect consumption insurance. Point (ii) states that aggregate state h, which is associated with a relatively high labour productivity for the employed, is not associated with a greater probability of facing a bad idiosyncratic shock (i.e., the probability of falling into unemployment, $1 - \alpha^s$, or to stay unemployed, ρ^s).[13]

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[11] Hence, employed agents are self-employed here. This specification is formally similar to one in which employed agents supply labour in a competitive labour market and interact with firms endowed with the constant-return-to-scale production function $y^{s_t} = z^{s_t} l$. Then, in equilibrium firms make no profit and employed agents' earn a wage equal to the marginal product of labour, z^{s_t}.

[12] Many authors, ranging from Mehra and Prescott (1985) to Alvarez and Jermann (2000), have analysed endowment economies that are stationary in growth rates, rather than in levels as we do. We adopt the level specification because our economy is a production one with endogenous labour supply and a period utility functional that is separable in consumption and leisure. In this context, stationarity in the growth rate of labour productivity is inconsistent with balanced growth, except for the somewhat specific case where the utility of consumption is logarithmic. In the separate technical appendix to the paper we analyse a growth-stationary variant of the model with log consumption utility that retains the main properties of our baseline level-stationary model.

[13] In short, this assumes that unemployment risk does not rise in a boom. Note that even in the case where

The date 0 distribution of agents over employment statuses is represented by a row vector $\omega_0 = [\omega_0^e \; \omega_0^u]$, where ω_0^e (ω_0^u) is the share of employed (unemployed) agents and $\omega_0^e + \omega_0^u = 1$. The laws of motion for the shares of employed and unemployed agents are given by, respectively,

$$\omega_t^e(s^t) = \alpha_t(s^t)\omega_{t-1}^e(s^{t-1}) + \left(1 - \rho_t(s^t)\right)\omega_{t-1}^u(s^{t-1}), \tag{2}$$

$$\omega_t^u(s^t) = \left(1 - \alpha_t(s^t)\right)\omega_{t-1}^e(s^{t-1}) + \rho_t(s^t)\omega_{t-1}^u(s^{t-1}). \tag{3}$$

3.3 Government

The government issues and rolls over riskless, zero-coupon bonds of various maturities that pay off one unit of the good at maturity. Bond maturities vary from 1 to $n \geq 1$, where n may be arbitrarily large. A bond of maturity $k > 1$ at date t becomes a bond of maturity $k - 1$ at date $t + 1$, and eventually yields one unit of the consumption good at date $t + k$. The date t price of this bond in terms of the consumption good is $p_{t,k}(s^t)$, and we define the price of a bond of maturity 0 by its payoff, i.e., $p_{t,0}(s^t) = 1$. The yield-to-maturity of a bond with maturity $k = 1, \ldots, n$ in history s^t is defined by the usual logarithmic expression, i.e., $r_{t,k}(s^t) = -k^{-1} \ln p_{t,k}(s^t)$.

There is no public consumption, so government outlays exactly equal the total payoff owed to the holders of bonds that are reaching maturity. At any date t, bonds issued at dates $t-1, t-2, \ldots$, $t - n$ with respective maturities 1, 2,..., n reach maturity. Bond payoffs are financed by both new bond issues and taxes. More specifically, at every date t, a quantity $A_{t,k}\left(s^t\right)$ of bonds paying 1 at date $t + k$ is issued at price $p_{t,k}(s^t)$, while a lump-sum tax on all *employed* agents, $\tau_t\left(s^t\right)$, is collected. Hence, the government budget constraint is:

$$\sum_{k=1}^{n} p_{t,k}\left(s^t\right) A_{t,k}\left(s^t\right) + \omega_t^e\left(s^t\right) \tau_t\left(s^t\right) = \sum_{k=1}^{n} A_{t-k,k}\left(s^{t-1}\right). \tag{4}$$

We restrict our attention to the case where the government issues the same quantity of k-period bonds in every period (i.e. $A_{t,k}\left(s^t\right) = A_k, \forall t \geq 0$), which implies that the overall quantity of bonds B_k of a given maturity k is also constant (i.e. $B_k = \sum_{j=0}^{n-k} A_{k+j}$, for any $k \geq 1$). The quantity B_k is composed of newly issued bonds of maturity k and of bonds issued earlier and coming closer to maturity. The lump sum tax paid by employed agents adjusts endogenously to satisfy (4) and is given by

$$\tau_t(s^t) = \frac{1}{\omega_t^e(s^t)} \sum_{k=1}^{n} \left(p_{t,k-1}(s^t) - p_{t,k}(s^t)\right) B_k, \tag{5}$$

As will become clear below, positing that only the employed are taxed in our baseline specification allows us to better isolate the liquidation risk premium on long bonds in the theoretical part of the paper. As we establish formally in the separate technical appendix, the effect of bond volumes on the yield curve are similar when the unemployed are also taxed.

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transition rates are constant (i.e., $\alpha^s = \alpha$ and $\rho^s = \rho$ for $s = l, h$) there will be some (procyclical) variations in idiosyncratic risk, because the labour income of the unemployed (δ) will be less cyclical than that of the employed.

3.4 Agents' behaviour

Each agent $i \in I$ has preferences over consumption and labour that are described by the subjective discount factor $\beta \in (0,1)$ and the instant utility function $u(c) - l$, where c is consumption, l labour supply and u is a $\mathcal{C}^2$ function satisfying $u'(.) > 0$, and $u''(.) < 0$ (this follows Scheinkmann and Weiss (1986)). Asset markets are incomplete, in that zero-coupon government bonds are the only assets that agents can trade. This implies that, first, there is no asset providing a payoff contingent on agents' idiosyncratic employment status (i.e., unemployment risk is uninsurable); and second, no agent can have negative asset wealth at any point in time.[14] We denote the quantity of k-period bonds held by agent i at the end of period t by $b_{t,k}^i$, and the corresponding bond holdings at the beginning of period 0 by $b_{-1,k}^i$ (specific assumptions about initial bond holdings will be made later on to avoid uninteresting transitory dynamics). Agent i's problem consists of choosing the sequences of consumption $c_t^i(s^t, e^{i,t})$, labour supply $l_t^i(s^t, e^{i,t})$, and bond holdings $(b_{t,k}^i(s^t, e^{i,t}))_{1 \le k \le n}$, defined over $S^t \times E^t$ to maximize expected intertemporal utility. From now on, we simplify notations by omitting the references to s^t and $e^{i,t}$ when no ambiguity arises. Agent i solves:

$$\max_{\{c^i, l^i, b^i\}} E_0^i \sum_{t=0}^{\infty} \beta^t \left(u\left(c_t^i\right) - l_t^i \right) \tag{6}$$

$$\text{s.t.} \quad c_t^i + \tau_t e_t^i + \sum_{k=1}^{n} p_{t,k} \, b_{t,k}^i = \sum_{k=1}^{n} p_{t,k-1} \, b_{t-1,k}^i + e_t^i z_t l_t^i + \left(1 - e_t^i\right)\delta, \tag{7}$$

$$b_{t,k}^i \ge 0, \quad k = 1, \ldots, n, \tag{8}$$

$$c_t^i, \ l_t^i \ge 0, \tag{9}$$

$$\lim_{t \to \infty} \beta^t u'\left(c_t^i\right) b_{t,k}^i = 0, \quad \text{for } k = 1, \ldots, n, \tag{10}$$

$$b_{-1,k}^i \ge 0, \quad k = 1, \ldots, n \text{ given}. \tag{11}$$

Equation (7) is agent i's budget constraint at date t: total wealth is made of the value of the bond portfolio inherited from last period's bond purchases, as well as wage income if the agent is employed (i.e., $e_t^i = 1$) or home production income if the agent is unemployed (i.e., $e_t^i = 0$); this wealth is used to purchase consumption goods, buy (or hold on to) bonds of various maturity, and pay taxes (if $e_t^i = 1$). The inequalities in (8) reflect the fact that private agents cannot issue bonds[15].

Conditions (9) and (10) are the non-negativity and transversality conditions –which are always satisfied in the equilibrium we consider–, and (11) is the agent's initial bond holdings.

Let $(\varphi_{t,k}^i)_{k=1,\ldots,n}$ be the Lagrange multipliers associated with the borrowing constraints in (8). These multipliers are positive functions defined over $S^t \times E^t$. The first-order conditions associated

---

[14] These properties are central in the literature on liquidity-constrained economies since the seminal work of Bewley (1980). See also Kehoe and Levine (2001), and the references therein.

[15] In the separate technical appendix we explore numerically a relaxed version of agents' problem wherein they face i) a nonzero debt limit on every maturity, and ii) a nonnegativity constraint on *total* wealth. We show that our main results regarding the impact of bond supplies on the shape of the yield curve go through in that scenario.

with the agent's program (6)–(10) are, for $k = 1, \ldots n$:

$$
\begin{cases}
u'\left(c_t^i\right) &= 1/z_t \quad & \text{if} \quad & e_t^i = 1, \\
l_t^i &= 0 \quad & \text{if} \quad & e_t^i = 0,
\end{cases}
\tag{12}
$$

$$
u'\left(c_t^i\right) p_{t,k} = \beta E_t \left[u'\left(c_{t+1}^i\right) p_{t+1,k-1} \right] + \varphi_{t,k}^i.
\tag{13}
$$

Equation (12) describes the agent's optimal labour supply. On the one hand, unemployed agents (i.e., for whom $e_t^i = 0$) do not supply labour. On the other hand, by the quasi-linearity of preferences, employed agents (i.e., for whom $e_t^i = 1$) work up to the point where their marginal utility of consumption (also equal to the marginal utility of wealth) is equal to $1/z_t$. As will become clear in Section 4.3 below, this will in turn imply that consumption and bond holdings are identical across employed agents –and hence independent of their history of idiosyncratic states. The Euler equation (13) sets the marginal cost of acquiring one unit of k-period bonds at date t equal to the marginal gain associated with its payoff at the next date. When the shadow cost of the borrowing constraint is positive, meaning that the constraint is binding ($\varphi_{t,k}^i > 0$), agent i would like to increase current consumption by issuing k-period bonds (but is prevented from doing so by assumption).

3.5 Market clearing and equilibrium

We denote by $\Lambda_t : \left(\mathbb{R}^+\right)^n \times E \to [0,1]$ the probability measure describing the distribution of agents across individual wealth (made of bonds of various maturities) and employment status in period t. For example, $\Lambda_t\left(b_1, \ldots, b_n, 1\right)$ denotes the measure of agents who are employed ($e_t^i = 1$) and hold a bond portfolio $b_1, \ldots, b_n$. This measure depends on the history of shocks and the initial distribution of agents, denoted Λ_0. The market-clearing conditions set the aggregate demand for bonds of each maturity equal to their exogenous supply, i.e.,

$$
\int_{(b_1, \ldots, b_k, \ldots, b_n, e) \in \left(\mathbb{R}^+\right)^n \times E} b_k \, d\Lambda_t\left(b_1, \ldots, b_k, \ldots, b_n, e\right) = B_k, \quad \forall k = 1, \ldots, n.
\tag{14}
$$

By Walras Law, the goods market clears when all bond markets clear.

Definition 1 (Equilibrium). *For an initial distribution of bond holdings $(b_{-1,k}^i)_{k=1,\ldots,n}$ and employment status Λ_0, an equilibrium consists of individual choices $\{c_t^i, (b_{t,k}^i)_{k=1,\ldots,n}, l_t^i\}_{t=0,\ldots,\infty}$ and bond prices $\{p_{t,k}\}_{k=1,\ldots,n,t=0,\ldots,\infty}$ such that:*

1. *Given prices, individual choices solve the agents' program (i.e., equations (6) to (10) hold);*

2. *Λ_t evolves consistently with individual choices and the transition matrices for individual and aggregate states;*

3. *All bond markets clear at all dates (i.e., equation (14) holds).*

4 Uninsured idiosyncratic risk, relative bond supplies, and the shape of the yield curve

This section investigates theoretically the impact of the supply of government bonds on the yield curve in our incomplete-market environment. In order to properly disentangle all the relevant effects at work, we proceed gradually by analysing three specifications of our framework. First, we characterise the yield curve with *complete* markets (Section 4.1). Second, we shut down insurance markets and study how the idiosyncratic risk affects the shape of the yield curve when bonds are in zero net supply –and hence no bond trades take place in equilibrium (Section 4.2). Finally, we show how the yield curve is affected under incomplete markets when agents trade positive net supplies of bonds (Section 4.3).

In all three economies, we focus on the equilibrium where bond prices only depend on the realisation of the current aggregate shock. From the literature on asset pricing with finite state-space (e.g., Mehra and Prescott (1985)), we conjecture (and verify) that bond prices at any date t only depend on the current aggregate state s_t, and not on the entire history s^t, i.e., $p_{t,k}(s^t) = p_{t,k}$, and we simply note p_k^s the price of a $k-$maturity bond in state $s \in \{h, l\}$. This price structure entails a form of stationarity, since bond prices depend only on their maturity and the current aggregate state.[16] In consequence, there are two *conditional* yield curves (one for each value of the aggregate state). To simplify the exposition, we keep the notation $p_{t,k}$ instead of p_k^s, when no ambiguity arises. The average (or 'unconditional') yield curve $(\bar{r}_k)_{k=1,...,n}$ is the weighted sum of conditional yields, where the weights are the unconditional frequencies of each aggregate state: $\bar{r}_k \equiv \eta^h r_k^h + \eta^l r_k^l$.

4.1 Complete markets

This section characterises the yield curve when insurance markets are complete for the idiosyncratic risk. In this situation, equation (12) implies that all agents have the same marginal utility of consumption at date t, i.e., $u'\left(c_t^i\right) = 1/z_t$ for all i. Then, from (13), bond prices satisfy for all $k = 1, \ldots, n$:

$$p_{t,k} = E_t \left[m_{t+1}^{CM} p_{t+1,k-1} \right] = E_t \left[\prod_{j=1}^{k} m_{t+j}^{CM} \right], \text{ where } m_{t+1}^{CM} = \beta \frac{z_t}{z_{t+1}} \tag{15}$$

is the agents' pricing kernel at date $t+1$. Iterating (15) forward, we find that $p_{t,k} = \beta^k E_t \left[z_t/z_{t+k} \right]$, so the yield-to-maturity at date t of a bond with maturity k is:

$$r_{t,k}^{CM} = -\ln(\beta) - \frac{1}{k} \left(\ln \left(E_t \left[z_{t+k}^{-1} \right] \right) - \ln(z_t^{-1}) \right). \tag{16}$$

---

[16] This stationarity is straightforward in both the complete-market and the incomplete market/zero net supply specifications. As we discuss further in Section 4.3 and Appendix D, in the incomplete market/positive net supply case it is tied to the limited cross-sectional heterogeneity that makes our analysis tractable.

The first term $-\ln\beta$ is simply the level of the interest rate without aggregate risk. The term $-\ln\left(E_t\left[z_{t+k}^{-1}\right]\right)-\ln(z_t^{-1})$ is the difference, expressed in terms of log-marginal utility, between the cost of purchasing the bond in the current period $(\ln(z_t^{-1})$ and its payoff at maturity, $\ln\left(E_t\left[z_{t+k}^{-1}\right]\right)$. This difference reflects the key motive for holding a k-period bond in the representative-agent framework, namely, that of smoothing consumption between date t and date $t+k$. The terminal payoff $(=1)$ occurs once in the bond lifetime, no matter the maturity of the bond. The yield premium $-k^{-1}\left(\ln\left(E_t\left[z_{t+k}^{-1}\right]\right)-\ln(z_t^{-1})\right)$ is therefore spread over the maturity of the bond and decreases (in absolute value) as the maturity of the bond rises –to eventually vanish at very long maturities. The property that aggregate risk affects short yields more than long yields will also be true in the incomplete-market environment studied below, for the same reason (although bondholders' pricing kernel and hence bond yields will of course differ from that in the complete-market case). The next proposition summarises the key properties of the yield curve under complete markets, and a discussion follows (all proofs are in the Appendix).[17]

Proposition 1 (Complete-market yield curves). *With complete markets,*

1. *Yields in both aggregate states converge towards a common limit $r_\infty^{CM}=-\ln\beta>0$ as $k\to+\infty$.*

2. *The yield curve in aggregate state h (l) lies below (above) r_∞^{CM} and is strictly increasing (decreasing) in k.*

3. *The average yield curve lies below r_∞^{CM} and is strictly increasing in k*

Part 1 of the proposition states that in the limit the long-run (i.e., infinite-maturity) yield on zero-coupon bonds with aggregate risk, r_∞^{CM}, is the same as the interest rate on all bonds in the absence of aggregate risk, i.e., the reward of time $-\ln\beta$. Part 2 applies to conditional yields at shorter maturities. Aggregate risk generates either a yield premium (in the bad state) or a yield discount (in the good state), whose value decreases with the maturity of the bond (and eventually reaches zero as $k\to+\infty$.) This reflects the demand for government bonds for hedging purposes against the aggregate risk. In the good state, the expected state at maturity is less favorable than the current state, and hence expected future marginal utility is greater than current marginal utility; this drives up the demand for bonds and produces a yield discount relative to the long-run rate. In the bad state, the opposite occurs: the expected state is more favorable than the current one and the demand for hedging is low, leading to yield premium and hence a reversion of the yield curve. This property, which will be valid for all model variants studied below, is a fairly general result coming from the bounded support of the aggregate shocks. Part 2 also states that the two conditional yield curves are monotonic, a property that follows from the assumed persistence of

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[17]Most of our results in Sections 4.1 (complete markets) and 4.2 (incomplete markets and zero net asset supplies) would hold under much more general preferences that those posited in Section 3.4. Both for consistency and clarity, we maintain our assumed preferences throughout.

aggregate shocks (as stated in Assumption A).[18] Finally, Part 3 pertains to the unconditional yield curve, i.e., the arithmetic average of the conditional curves weighted by their frequency of occurrence. For every yield, the yield discount in the good state is on average greater than the yield premium in the bad state, as hedging is cheaper in the good state (where marginal utility is low) than in the bad state (where marginal utility is high). Hence, the two do not cancel out, resulting in a positive average discount on short bonds. Given the assumed persistence of aggregate shocks and the fact that the terminal payoff is spread over the life of the bond, the average yield curve is itself monotonic.

I.i.d. example. For the complete-market economy as well as the following next two economies with incomplete markets, we provide a specific example of yield curve under i.i.d. aggregate shocks, in which case explicit expressions for yields at the two ends of the curve can be obtained throughout. Namely, we assume that $z^l = 1 - \varepsilon$ and $z^h = 1 + \varepsilon$, with $\pi^h = \pi^l = 1/2$ and ε small. From (16), the yield on one-period bonds in state $s = l, h$ is then $r_1^{CM,s} = -\ln(\beta) - \left(\ln\left(0.5\left[1/z^h + 1/z^l\right]\right) - \ln(1/z^s)\right)$, and a second-order approximation to the average yield, $\bar{r}_1^{CM} = (r_1^{CM,l} + r_1^{CM,h})/2$ around $\varepsilon = 0$ gives:

$$\bar{r}_1^{CM} = -\ln\beta - \frac{1}{2}\varepsilon^2. \tag{17}$$

Given r_∞^{CM}, the slope of the yield curve under complete markets, $\Delta^{CM} = r_\infty^{CM} - \bar{r}_1^{CM}$, is

$$\Delta^{CM} = \frac{1}{2}\varepsilon^2. \tag{18}$$

4.2 Incomplete markets and zero net bond supply

Our second benchmark economy features incomplete markets against the idiosyncratic risk and government bonds in zero net supply for all maturities (i.e., for all $k = 1, \ldots, n$, $A_k = 0$, so that $B_k = 0$). Consequently, the taxes in (5) are also equal to zero at all dates. Since no agent is allowed to have a short position in bonds (by (8)), bond prices must adjust at all dates up to the point where the agents with the highest marginal valuation of the bonds have net demands equal to zero, and no trade ever takes place between the agents. The following assumption allows us to focus on the interesting case, where uninsured idiosyncratic income risk results in endogenously incomplete participation in bond markets.

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[18]To see intuitively why persistence of aggregate states is required for monotonicity, consider the extreme opposite case where the aggregate state oscillates in a deterministic fashion, i.e., $\pi^h = \pi^l = 0$, and suppose that the state at date t is h (a similar reasoning applies in state l). It straightforward, using (15), to show that the resulting yield curve converges to $-\ln\beta$ in an oscillatory manner (i.e., $r_2^{CM,h} > r_1^{CM,h}$, $r_3^{CM,h} < r_2^{CM,h}$, $r_4^{CM,h} > r_3^{CM,h}$ etc.). The reason is as follows. The date t demand for one-period bonds is high (because the state at date $t+1$ is l for sure), leading to a low value of r_1^h. The demand for two-period bonds is low (because the state at $t+2$ is h for sure), leading to a high value of r_2^h. However, the demand for three-period bonds is high (because the state at $t+3$ is l for sure), causing the value of r_3^h to fall below r_2^h, and so forth. Under assumption A, such oscillations never occur, whatever the values of z^l and z^h.

Assumption C (Participation condition).

$$\frac{\alpha^l/z^h + \left(1 - \alpha^h\right) u'\left(\delta\right)}{1/z^l} > \frac{\left(1 - \rho^h\right)/z^l + \rho^l u'\left(\delta\right)}{u'\left(\delta\right)}. \tag{19}$$

The left hand side of (19) is a lower bound on the pricing kernel of employed (i.e., high-income) agents, while the right hand side is an upper bound on the pricing kernel of unemployed (i.e., low-income) agents. Hence, this assumption will ensure that the pricing kernel of the employed is always greater than that of the unemployed, implying that only the former buy bonds and are making prices. A sufficient condition for (19) to hold is that δ be sufficiently small, i.e., that the unemployed be sufficiently worse-off than the employed.[19]

The following proposition characterizes the bond price structure in the zero-volume economy, where bond prices are derived from the Euler equations (13).

Proposition 2 (Pricing kernel decomposition under zero net supply). *There exists a unique equilibrium such that:*

1. *Bond prices are given by $p_{t,k} = E_t\left[m_{t+1}^{ZV} p_{t+1,k-1}\right] = E_t\left[\prod_{j=1}^{k} m_{t+j}^{ZV}\right]$, where the pricing kernel m_{t+1}^{ZV} can be written as*

$$m_{t+1}^{ZV} = m_{t+1}^{CM} I_{t+1}^{ZV}, \quad with \quad I_{t+1}^{ZV} = \frac{\alpha_{t+1}/z_{t+1} + \left(1 - \alpha_{t+1}\right) u'(\delta)}{1/z_{t+1}} \; (\geq 1); \tag{20}$$

2. *The yield curves in states h and l converge towards a common, constant limit r_∞^{ZV};*

3. *If $\alpha^h - \alpha^l$ is sufficiently small, the yield curve in state h (state l) lies strictly below (above) r_∞^{ZV} and is monotonically increasing (decreasing) in the maturity k.*

Part 1 of the proposition states that the pricing kernel in the incomplete-market case with zero net supply, m_{t+1}^{ZV} (where "ZV" stands for "zero volume"), is given by the pricing kernel in the complete-market case, m_{t+1}^{CM} in (15), *times* an upward proportional bias I_{t+1}^{ZV} reflecting the possibility of being hit by an uninsurable unemployment shock in the next period –in which case future marginal utility is $u'(\delta)$ rather than $1/z_{t+1}$.[20] While m_{t+1}^{CM} is determined by agents' willingness to hedge the aggregate risk, I_{t+1}^{ZV} reflects their willingness to hedge the idiosyncratic risk. Parts 2 and 3 follow from the general properties of m_{t+1}^{ZV} and parallel some similar results obtained

---

[19] The pricing kernel of the employed is $\beta\left(\alpha_{t+1}/z_{t+1} + \left(1 - \alpha_{t+1}\right) u'\left(\delta\right)\right)/\left(1/z_t\right)$ which is never less than the left hand side of (19) for all $(z_t, z_{t+1}) \in \{z^l, z^h\}^2$. Symmetrically, the pricing kernel of the unemployed is $\beta\left(\left(1 - \rho_{t+1}\right)/z_{t+1} + \rho_{t+1} u'\left(\delta\right)\right)/u'\left(\delta\right)$, which is never greater than the right hand side of (19) for all $(z_t, z_{t+1}) \in \{z^l, z^h\}^2$. Condition (19) is not strong and will be satisfied under any plausible parameterisation of the extent of unemployment risk (as given by α^s and ρ^s, $s = l, h$) and direct unemployment insurance (as determined by δ).

[20] Constantinides and Duffie (1996) exhibit a similar factorisation of the pricing kernel under incomplete markets in the case where agents face *permanent* rather than transitory idiosyncratic shocks. Krueger and Lustig (2010) also use a pricing kernel factorisation and exhibit a set of conditions under which incomplete markets do not to affect the equity premium in their endowment economy. Note that none of their conditions hold in our production economy, and that our focus is on the pricing of non-contingent bonds (rather than a Lucas tree).

in the complete-market case studied above. In particular, the long-run yield (i.e., on infinite-maturity bonds) r_∞^{ZV} is not conditional on the aggregate state –but generally differs from that in the complete-market case, r_∞^{CM} (see Appendix B for details).

Before we further characterise the yield curve in the incomplete-market, zero-volume case, let us briefly discuss some implications of Proposition 2. First, incomplete insurance against unemployment risk, as summarized by a lower value of α_{t+1}, tends to raise I_{t+1}^{ZV} and thus to exert a downward pressure on both conditional yield curves –and, by implication, on the average curve. This is a mere reflection of the demand for bonds for self-insurance purposes, which rises as idiosyncratic income risk rises, all else equal. Second, holding α_{t+1} constant, the conditional covariance between the two components of the pricing kernel is *negative*. On the one hand, given z_t, a higher value of z_{t+1} lowers agents' willingness to hedge the aggregate risk, so that m_{t+1}^{CM} falls (since expected future marginal utility, $1/z_{t+1}$, falls relative to current marginal utility, $1/z_t$.) On the other hand, a higher value of z_{t+1} raises the difference in marginal utility between being employed $(1/z_{t+1})$ and being employed $(u'(\delta))$. This raises agents' willingness to hedge the idiosyncratic risk, so that I_{t+1}^{ZV} rises.[21] This negative correlation reduces the expected value of the pricing kernel $m_{t+1}^{CM} I_{t+1}^{ZV}$, thereby contributing to reducing prices and increasing yields.

Proposition 3 below summarises how idiosyncratic volatility and bond supplies affect the shape of the yield curve.

Proposition 3 (Incomplete-market, zero-net supply yield curves)**.** *Assuming that $z^h - z^l$ is small, then the yield curve has the following properties:*

1. *Without aggregate shocks (i.e. $z^s = z = 1$ and $\alpha^s = \alpha < 1$, $s = l, h$), the yield curve is flat and its level is $r_\infty^{CM} - \ln\left(\alpha + (1-\alpha)zu'(\delta)\right) < r_\infty^{CM}$, which increases with α.*

2. *When the aggregate state affects productivity (i.e., $z^h > z^l$) but not the probability of a bad idiosyncratic shock (i.e., $\alpha^h = \alpha^l = \alpha$), an increase in that probability (that is, a fall in α) i) lowers the level of the yield curve in both states l and h, and ii) lowers the level and the slope of the average yield curve.*

3. *Time-variations in the probability of a bad idiosyncratic shock (i.e., $\alpha^h > \alpha^l$) may either raise or lower the slope of the average yield curve, relative to both the complete-market case and the incomplete-market case with constant transition probabilities. Given a pair of aggregate productivity levels (z^h, z^l), there always exists a pair of individual transition rates (α^h, α^l) such that the slope of the average yield curve is larger under incomplete markets than under complete markets.*

Part 1 of the proposition states that in an economy with idiosyncratic risk but no aggregate shocks, the yield curve is flat and its level goes down as unemployment risk becomes more severe

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[21] This negative co-movement between m_{t+1}^{CM} and I_{t+1}^{ZV} will prevail in any economy in which the income earned when unemployed is less sensitive to the aggregate state than that earned when employed (e.g., because unemployment benefits are less cyclical than wages.)

(i.e., α decreases). Intuitively, a higher level of idiosyncratic risk raises the willingness to hedge that risk (formally $I_{t+1}^{ZV} = I^{ZV} = \alpha + (1-\alpha)\,zu'(\delta)$ is constant and decreasing in α); hence, the demand for bonds for self-insurance purpose rises, which lowers the equilibrium real interest rate (since $r_\infty^{ZV} = r_\infty^{CM} - \ln I^{ZV}$). This effect of idiosyncratic risk on the steady-state interest rate is similar to that at work in Huggett (1993) or Aiyagari (1994), among many others.

Part 2 analyses the case where the aggregate state affects labour productivity but not the extent of unemployment risk.[22] First, the levels of the conditional yield curves fall as α does, in either aggregate state (for the same reason as in Part 1). Second, an increase in idiosyncratic risk (i.e., a fall in α) lowers the slope of the yield curve, i.e., it lowers long yields more than short yields. The reason for this is that, as explained above, the demands for hedging the aggregate and the idiosyncratic risk go in opposite direction, i.e., the conditional covariance between m_{t+1}^{ZV} and I_{t+1}^{ZV} is negative. This covariance increases in absolute value as α falls, which drives down the demand for bonds and hence reduces bond prices and raises bond yields. This effect is higher for short bonds than for long bonds because aggregate risk affects short yields more than longer yields, for the reason discussed in Section 4.1. The i.i.d. example below provides a simple expression for the interaction term due to the correlation of the two components of the pricing kernels.

Part 3 of the proposition examines how time-variations in the probability of a bad idiosyncratic shock may alter the shape of the yield curve, relative to the complete-market case. To see why the effect is ambiguous in general, consider (with no loss of generality) the impact of a mean-preserving spread in (α^h, α^l) relative to the case where α is constant. On the one hand, the higher value of α^h lowers the demand for self-insurance in the good state; conversely, the lower value of α^l raises the demand for self-insurance in the bad state. The overall impact on the average yield curve is ambiguous in general and depends on the persistence of the aggregate states. In particular, one may easily construct example in which a time-varying probability of a bad idiosyncratic shock leads to a high average term premium on long bonds –in the same way as Mankiw (1986) reported that it may generate a high risk premium. It is notably the case under i.i.d. aggregate shocks, as we now show.

I.i.d. example. Again, we use the i.i.d. example to illustrate how the presence of uninsured idiosyncratic risk alters the shape of the yield curve relative to the complete-market case analysed above. Expressions (C.1)–(C.2) in Appendix C provide the second-order approximations to the short and long yields for a general transition matrix across aggregate states, T. The following formulas are obtained by imposing the i.i.d. shock structure ($\pi^h = \pi^l = 1/2$, $z^h = 1 + \varepsilon$ and $z^l = 1 - \varepsilon$) to those expressions. When aggregate risk only affects z but not α, the yield on long

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[22] The assumption that $z^h - z^l$ is small allows us to use second-order Taylor expansions of the relevant pricing kernels around the steady state and thereby to characterise analytically the impact of those variables on the slope of the yield curve.

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(infinite-maturity) bonds is given by:

$$r_\infty^{ZV} = r_\infty^{CM} \underbrace{- \ln\left(\alpha + (1-\alpha)u'(\delta)\right)}_{\text{Demand for self-insurance}}. \tag{21}$$

As in the complete-market case, aggregate volatility leaves the yield of long bonds independent of the aggregate state. The additional implication of i.i.d. shocks is that the extent of aggregate volatility –as measured by ε– has no impact on the level of this common long yield; only the (constant) probability of a bad idiosyncratic shock (given by $1 - \alpha$) does, a reflection of agents' higher demand for bonds for self-insurance purpose. The average yield on one-period bonds, $\bar{r}_1^{ZV} = \sum_{s=l,h} r_1^{ZV,s}/2$, can be decomposed as follows:

$$\bar{r}_1^{ZV} = \bar{r}_1^{CM} \underbrace{- \ln\left(\alpha + (1-\alpha)u'(\delta)\right)}_{\text{Demand for self-insurance}} + \underbrace{\frac{(1-\alpha)u'(\delta)}{\alpha + (1-\alpha)u'(\delta)}\varepsilon^2}_{\text{Interaction term}}. \tag{22}$$

The short yield is the sum of three terms. The first, $\bar{r}_1^{CM}$, is the short yield under complete markets, which is recovered in the incomplete-market case when there is no idiosyncratic shocks (i.e., $\alpha = 1$). The second term is the equilibrium yield discount coming from the demand for hedging the idiosyncratic risk, or "demand for self-insurance", which affects the short yield and the long yield equally. The last term in (22) reflects the interaction between the demands for hedging the aggregate and the idiosyncratic risks.[23] As discussed above, when α is not time-varying the conditional covariance at date t between m_{t+1}^{CM} and I_{t+1}^{ZV} induced by stochastic changes in z_{t+1} is negative. This implies that idiosyncratic risk tends to mitigate agents' demand for hedging the aggregate risk, which translates into lower equilibrium prices (that is, higher equilibrium yields).

From (21)–(22), the slope of the yield curve in the incomplete-market, zero-volume case, $\Delta^{ZV} = r_\infty^{ZV} - \bar{r}_1^{ZV}$, is

$$\Delta^{ZV} = \Delta^{CM} - \underbrace{\frac{(1-\alpha)u'(\delta)}{\alpha + (1-\alpha)u'(\delta)}\varepsilon^2}_{\text{Interaction term}} \tag{23}$$

The first term in the right hand side of (23) is the slope of the yield curve under complete markets. The second term comes from the interaction of aggregate and idiosyncratic risks. Since, as discussed above, the interaction term raises the short yield without affecting the long yield, an increase in idiosyncratic risk (that is, a fall in α holding ε constant) lowers the slope of the yield curve. It follows that the slope of the yield curve in the incomplete-market, zero-volume case cannot be higher than the slope under complete markets –as long as the probability of a bad idiosyncratic shock is not itself time-varying and bonds are in zero net supply.

Time-varying probability of a bad idiosyncratic shock. Let us now illustrate the last point of Proposition 3 in the context of our i.i.d. example. To this purpose, we introduce (without loss of

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[23] As mentioned above, the size of interaction term is scaled by the inverse of the maturity (k^{-1}). This is why it shows up in the expression for the one-period yield in (22), but not in that for the very long yield in (21).

generality) a mean preserving spread in α, i.e., $\alpha^h = \alpha + a$ and $\alpha^l = \alpha - a$, where a is positive but small. From equations (21)–(22) above and (C.1)–(C.2) in Appendix C, the yields at the two ends of the curve become:

$$\tilde{r}_1^{ZV} = \bar{r}_1^{ZV} + \frac{1}{\alpha + (1 - \alpha)u'(\delta)}a\varepsilon, \quad \tilde{r}_\infty^{ZV} = r_\infty^{ZV} + \frac{u'(\delta)}{\alpha + (1 - \alpha)u'(\delta)}a\varepsilon.$$

In the i.i.d. case, times variations in α $(a > 0)$ shift the entire yield curve upwards, provided that labour productivity is also affected $(\varepsilon > 0)$. Moreover, since $u'(\delta) > 1$ (by Assumption B), such variations shift the long yield more than the short yield. It follows that the slope for the yield curve under time-varying idiosyncratic risk is given by:

$$\tilde{\Delta}^{ZV} = \Delta^{ZV} + \frac{u'(\delta) - 1}{\alpha + (1 - \alpha)u'(\delta)}a\varepsilon.$$

Using this expression and that for Δ^{ZV} above, we infer that $\tilde{\Delta}^{ZV} > \Delta^{CM}$ –i.e., the slope of the yield curve is larger under incomplete markets than under complete markets– provided that

$$a > \frac{(1 - \alpha)u'(\delta)}{u'(\delta) - 1}\varepsilon > 0,$$

that is, the impact of the aggregate state on the probability of a bad idiosyncratic shock must be sufficiently large compared to its effect on labour productivity.

While the analysis is more involved away from the i.i.d. case, the combined effects of aggregate and idiosyncratic uncertainty on the yield curve are similar. In particular, i) the demand for hedging the idiosyncratic shocks affects all yields, while the demand for hedging the aggregate risk only affects short yields; ii) with constant probability of a bad idiosyncratic shock $1 - \alpha$, an increase in this probability lowers long yields more than short yields; and iii) there always exist $a = (\alpha^h - \alpha^l)/2$ such that the slope of the yield curve is larger under incomplete markets than under complete markets.

4.3 Incomplete markets and positive net bond supply

We now move away from the zero-net-supply assumption and investigate the impact of the volume of bonds on the shape yield curve. As discussed above, the incompleteness of insurance markets implies that some agents (the employed) express a specific demand for bonds in order to hedge the idiosyncratic risk they face (in addition to attempting to hedge the aggregate risk). When bonds are in positive net supply, these agents are effectively able to use bonds as a buffer against idiosyncratic shocks, thereby partially alleviating the lack of fully functioning credit and insurance markets. This has two key implications. First, the entire yield curve is affected by the net supply of bonds, since the latter affect agents' ability to hedge the idiosyncratic risk. Second, bonds are effectively traded in equilibrium: on the one hand, the agents who are hit by a bad idiosyncratic income shock wish to sell some bonds in order to partly insulate their consumption from the shock; on the other hand,

the agents who are not yet hit by the shock (but anticipate that they might be in the future) are willing to buy those bonds for self-insurance. That is, the repeated occurrence of idiosyncratic shocks together with agents' desire to hedge them the best they can generate cross-agents trade. In particular, agents may find themselves in a situation where they have to liquidate long-maturity bonds –due to the occurrence of a bad idiosyncratic shock, which causes the borrowing constraint to start binding– precisely at a time when their trading price is low –because the aggregate state is itself unfavourable. As we show, this generates a specific "liquidation risk premium" on long bonds, the value of which depends on the quantity of such bonds in agents' equilibrium portfolios.

As discussed in the introduction, a major issue with incomplete-market models with positive net asset supplies is their lack of tractability. In particular, these models typically generate a large-dimensional cross-sectional distribution of wealth, which must be approximated numerically and solved jointly with the agents' optimal asset holding decisions. This in turn drastically limits the number of assets that these models can handle. To circumvent this issue –and thereby price the entire yield curve, including the price of bonds with very long maturity–, our approach in this paper is to impose some restrictions on the structure and deep parameters of the model so that the model endogenously generates a cross-sectional distribution with a small number of wealth states. To be more specific, the tractability of our framework is the outcome of two underlying assumptions. First, quasi-linear preferences imply that all employed agents are willing to work as much as necessary to bring their marginal utility of consumption (equal to their marginal utility of wealth) to $1/z$ (see (12)). Consequently, all employed agents share the same consumption level, regardless of their history of idiosyncratic shocks. Second, the transitory nature of idiosyncratic shocks and the assumption that bonds are in positive but *small* supplies implies that agents in the bad idiosyncratic state would be willing to *issue* bonds against future income –but are prevented from doing so by the borrowing constraint. As we show in Appendix D, taken together these two properties imply that i) all employed agents share the same pricing kernel and hold the same (end-of-period) asset wealth, regardless of their employment history; and ii) all unemployed agents face a binding borrowing constraint (and hence hold no asset at the end of the period), regardless of their employment history. Consequently, our model generates a two-state cross-sectional distribution of wealth as an equilibrium outcome, which allows us to study bond pricing and trading analytically for an arbitrarily large number of maturities.

For simplicity, we focus on the case where transition probabilities across employment statuses are constant –in addition to assuming that bond supplies are small.

Assumption D (Small bond volume and constant probability of a bad idiosyncratic shock). *(i)* A_k, $k = 1, \ldots, n$ *is small; (ii)* $\alpha^l = \alpha^h = \alpha < 1$.

The following proposition generalises our pricing-kernel decomposition for the case where markets are incomplete and bonds are in positive net supply.

Proposition 4 (Pricing kernel decomposition under positive net supply). *There exists a unique equilibrium such that:*

1. *All employed agents buy the same amount of bonds of each maturity, while all unemployed agents face a binding borrowing constraint (and consequently hold no bonds);*

2. *The date t price of a bond of maturity k is $p_{t,k} = E_t\left[m_{t+1}^{PV} p_{t+1,k-1}\right] = E_t \prod_{j=1}^{k} m_{t+j}^{PV}$, where*

$$m_{t+1}^{PV} = m_{t+1}^{CM} I_{t+1}^{PV} \quad and \tag{24}$$

$$I_{t+1}^{PV} \equiv \frac{\alpha/z_{t+1} + (1-\alpha)\,u'\left(\delta + \sum_{j=1}^{n} p_{t+1,j-1}\left(B_j/\omega^e\right)\right)}{1/z_{t+1}}. \tag{25}$$

The first point of the proposition states the homogeneity properties that make our analysis tractable. First, the equilibrium features "full asset liquidation", in the sense that agents hit by an unemployment shock liquidate their entire portfolio instantaneously (formally, $b_{t,k}^i = 0$ for all $k = 1, \ldots, n$ if $e_t^i = 0$). Second, bond holdings are identical across employed agents (i.e., $b_{t,k}^i = b_{t,k} > 0$ for all k if $e_t^i = 1$)[24].

Under assumption D, employed agents are in constant number $\omega^e = (1-\alpha)/(2-\alpha-\rho)$, so the relevant market clearing conditions imply that $b_j = B_j/\omega^e$, for $j = 1, \ldots, n$: all agents hold positive bond quantities and there is no short position at any maturity.

The second point of the proposition generalises the pricing kernel decomposition of the previous section to account for the fact that employed agents now do hold bonds in equilibrium. Like in the zero-net-supply case, the equilibrium pricing kernel is the product of the complete-market kernel, m_{t+1}^{CM}, and a correction reflecting the demand for hedging the idiosyncratic risk, I_{t+1}^{PV} (where "PV" stands for "positive volume"). The key difference with the zero-net-supply case is that bond volumes now enter the correction term I_{t+1}^{PV} (symmetrically across bondholders, by point 1.) Indeed, in an economy with positive net bond supplies, the portfolio held by the agents may be liquidated when a bad idiosyncratic shock hits, thereby limiting the associated drop in individual consumption. The term I_{t+1}^{PV} reflects these better self-insurance possibilities via a lower marginal utility in case a bad idiosyncratic shock hits (the $u'(.)$ term in (25)).

We are now in a position to state our main comparative-static results regarding the way relative bond supplies affect the shape (i.e., the level and the slope) of the yield curve.

Proposition 5 (Incomplete-market, positive-net supply yield curves)**.** *Assume that $z^h - z^l$ is small. Then,*

1. *An increase in the supply of bonds of any maturity i) raises the level of the yield curve in both states l and h, and ii) raises the level and the slope of the average yield curve;*

2. *An increase in the supply of long bonds with maturity $k \geq 2$ raises more the slope of the average yield curve than an increase in the supply of one-period bonds.*

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[24]Under our specification for the borrowing constraint (see (8)), only public debt enters the portfolio liquidation value. It would not necessarily be the case if the borrowing constraint were on the total value of the portfolio (i.e., $\sum_{k=1}^{n} p_{t,k}\, b_{t,k}^i \geq 0$). In this case, agents could use long positions in some maturities to back short positions in other maturities. We study such a configuration numerically in the separate technical appendix.

The intuition for the latter two results follow from equations (24)–(25) which, despite the fact that they do not as such explicitly solve for equilibrium bond prices and yields –since the bond prices $p_{t,j}$ enter both sides of the asset-pricing formula–, nevertheless provide useful insights about their determinants (see the proof in Appendix D for an explicit solution for bond prices and yields). The impact of bond supplies on the *level* of the curve results from two effects. First, by raising the supply of aggregate liquidity that agents may hold in equilibrium, a greater supply of bonds of any maturity facilitates self-insurance and reduces the (marginal utility) cost associated with a bad idiosyncratic shock. The demand for hedging the idiosyncratic risk becomes more satiated (due to decreasing marginal utility), which depresses all prices and raises all yields in both aggregate states. Second, higher bond holdings of any maturity except one-period bonds raise the volatility of the liquidation value of the portfolio, $\sum_{j=1}^{n} p_{t+1,j-1} \left(B_j/\omega^e \right)$, and hence that of the pricing-kernel correction I_{t+1}^{PV} (one-period bonds do not affect this volatility since they pay out 1 for sure in the next period). This "portfolio volatility" effect raises the consumption risk associated with holding bonds, which feeds back to the entire yield curve and raises yields at all maturities.

The impact of bond supply and their maturity on the *slope* of the yield curve also results from two effects. First, the size of the portfolio $\sum_{j=1}^{n} p_{t+1,j-1} \left(B_j/\omega^e \right)$ modifies the correlation between the two components of the pricing kernel, m_{t+1}^{CM} and I_{t+1}^{PV}, and hence the value of the interaction term discussed in Section 4.2. Formally, for any given value of α, a greater value of $\sum_{j=1}^{n} p_{t+1,j-1} \left(B_j/\omega^e \right)$ lowers the marginal utility term $u'(.)$ in (25). Following an increase in z_{t+1} –which moves m_{t+1}^{CM} downwards–, the rise in I_{t+1}^{PV} is smaller than when $\sum_{j=1}^{n} p_{t+1,j-1} \left(B_j/\omega^e \right) = 0$. Hence, the negative correlation between m_{t+1}^{CM} and I_{t+1}^{PV} is smaller, which mitigates the impact of the interaction term on yields. This effect is at work after an increase in the quantity of bonds of any maturity, including one-period bonds (as stated in Point 1 of the proposition).

Second, the slope of the yield curve is affected by the *relative* supply of bonds, a direct impli-cation of the *liquidation risk* associated with holding bonds with maturity greater than one. This risk refers to the risk that the bonds must be liquidated –if the holder is hit by a bad idiosyncratic shock– precisely at a time when the resale price of the bond is low –because the aggregate state is itself unfavourable. Of course, long bonds may also have to be liquidated in good times but, because marginal utility is decreasing, these potential gains do not offset the cost of having to sell in bad times, so on average the premium on long bonds must be positive. Formally, the liquidation risk results from the covariance between the liquidation value of the portfolio, $\sum_{j=1}^{n} p_{t+1,j-1} \left(B_j/\omega^e \right)$, and bond prices, $p_{t+1,k}$, both of which fluctuate with the aggregate shock. This covariance is zero for one-period bonds –as the latter pay $p_{t+1,0} = 1$ for sure in the next period–, so such bonds do not command a liquidation risk premium in equilibrium. For longer bonds, and provided that $\alpha < 1$ (so that agents effectively face idiosyncratic income variations), the co-movements between the portfolio's liquidation value and bond prices generates a negative covariance term between the pricing kernel m_{t+1}^{PV} –via the marginal utility term in (25)– and the bonds' future resale price. This negative covariance depresses the current price of such bonds, i.e., it raises their yield. In short, by holding long rather than short bonds, agents run the risk of finding themselves poor precisely

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when they most need cash –and thus require a compensation for bearing this risk.

I.i.d example. Again, we provide further intuition for our results by means of a second-order approximation to the yield curve under small, i.i.d. aggregate shocks. Let $\bar{p}_k$ denote the price of a k-period bond in the steady state (without aggregate shocks) and $\overline{W} = \sum_{k=1}^{\infty} \bar{p}_k b_k$ the corresponding value of bond holders' portfolio, where $b_k = B_k/\omega^e$ is the equilibrium quantity of k-period bonds in any bondholder's portfolio. By assumption, $\overline{W}$ is small because volumes are small, and an increase in the volume of bonds of any maturity raises $\overline{W}$. In this case, the approximate expressions for the long and the short yields, are given by (see equations (F.7)–(F.8) in Appendix F):

$$r_\infty^{PV} = r_\infty^{ZV} + \underbrace{\frac{(1-\alpha)(-u''(\delta))}{\alpha + (1-\alpha)u'(\delta)}\overline{W}}_{\text{Impact of volumes on demand for self-insurance}} + \underbrace{\frac{\alpha(1-\alpha)(-u''(\delta))}{(\alpha + (1-\alpha)u'(\delta))^2}\left(\overline{W} - b_1\right)\varepsilon^2}_{\text{Portfolio volatility}}$$

$$+ \underbrace{\frac{(1-\alpha)(-u''(\delta))}{\alpha + (1-\alpha)u'(\delta)}\left(\overline{W} - b_1\right)\varepsilon^2}_{\text{Liquidation risk}} \tag{26}$$

$$\bar{r}_1^{PV} = \bar{r}_1^{ZV} + \underbrace{\frac{(1-\alpha)(-u''(\delta))}{\alpha + (1-\alpha)u'(\delta)}\overline{W}}_{\text{Impact of volumes on demand for self-insurance}} + \underbrace{\frac{\alpha(1-\alpha)(-u''(\delta))}{(\alpha + (1-\alpha)u'(\delta))^2}\left(\overline{W} - b_1\right)\varepsilon^2}_{\text{Portfolio volatility}}$$

$$- \underbrace{\frac{\alpha(1-\alpha)(-u''(\delta))}{(\alpha + (1-\alpha)u'(\delta))^2}\overline{W}\varepsilon^2}_{\text{Impact of volumes on interaction term}} \tag{27}$$

The latter two equations summarise how bond volumes affect bond yields at the two ends of the curve, relative to the zero-net-supply, incomplete-market yield curve. As discussed above, the two yields under consideration incorporate two additional level effects –i.e., affecting both yields equally–, relative to their zero-net-supply counterparts. First, an increase in the supply of any bond facilitates self-insurance, which lowers the demand for bonds and hence raises all yields (the "Impact of volumes on demand for self-insurance" term in (26)–(27)).[25] Second, an increase in bond holdings with maturity greater than one raises the volatility of bondholders' portfolio (and hence that of the pricing kernel), which deters additional bond purchases and again raises *all* yields (hence the "Portfolio volatility" term in (26)–(27)). Again, this second level effect does not apply to holdings of one-period bonds, since those generate no portfolio volatility; hence, a pure increase in the supply of short bonds (such that $\mathrm{d}\overline{W} = \mathrm{d}b_1$) only shifts the level of the yield curve via the

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[25] To see most clearly how positive bond volumes impact the demand for self-insurance, consider how the term $-\ln\left(\alpha + (1-\alpha)u'(\delta)\right)$ in equations (21)–(22) is modified when the value of the liquidated portfolio, $\overline{W}$, is added to the consumption of agents hit by a bad idiosyncratic shock –so that the marginal utility term is $u'(\delta + \overline{W})$ instead of $u'(\delta)$. Then, for $\overline{W}$ small we have

$$-\ln\left(\alpha + (1-\alpha)u'(\delta + \overline{W})\right) \simeq \underbrace{-\ln\left(\alpha + (1-\alpha)u'(\delta)\right)}_{\text{Demand for self-insurance (zero net supply)}} + \underbrace{\frac{(1-\alpha)(-u''(\delta))}{\alpha + (1-\alpha)u'(\delta)}\overline{W}}_{\text{Impact of volumes on demand for self-insurance}}$$

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demand for self-insurance channel.

From (26)–(27), the slope of the yield curve is given by:

$$\Delta^{PV} = \Delta^{ZV} + \underbrace{\frac{\alpha(1-\alpha)(-u''(\delta))}{(\alpha + (1-\alpha)u'(\delta))^2}\overline{W}\varepsilon^2}_{\text{Impact of volumes on interaction term}} + \underbrace{\frac{(1-\alpha)(-u''(\delta))}{\alpha + (1-\alpha)u'(\delta)}\left(\overline{W} - b_1\right)\varepsilon^2}_{\text{Liquidation risk premium}}. \qquad (28)$$

Positive bond volumes affect the slope (relative to the zero-net-supply case) by i) lowering the short yield (see the "Impact of volumes on interaction term" in (27)), and ii) raising the long yield –provided that the supply of bonds with maturity greater than one is raised (see the "Liquidation risk premium" term in (26)). As discussed above, the impact on the slope coming from a lower short yield relative to the zero-volume case comes from the fact that the interaction term is now affected by the supply of bonds.[26] The last term in (28), isolates the liquidation risk premium commanded by long bonds over short bonds. This liquidation risk only applies to the part of agents' wealth made of bonds with maturity greater than one, since the liquidation value of one-period bonds is independent of the aggregate state. To summarize, equation (28) implies that an increase in the supply of bonds of any maturity raises the slope of the yield curve, and an increase in the supply of bonds with maturity greater than one raises more the slope than an increase in the supply of one-period bonds (due to the liquidation risk premium). Hence, both the size and the maturity structure of the debt affect the slope of the yield curve.

Numerical application.   Let us illustrate the effect of bond volumes on the shape of the yield curve by means of a simple numerical example. The period is a year. We assume that $u(c) = Ac^{1-\sigma}/(1-\sigma)$, with $\sigma = 2.5$ –a standard value for the level of risk aversion– and $A = 0.4$ –a scaling parameter set to the low productivity state (see below). We set β so that, given σ, the average one-year real interest rate r_1 match its empirical counterpart ($= 1.8\%$);[27] this gives $\beta = 0.967$. The transition probabilities across aggregate states are set to $\pi^h = 0.8$ and $\pi^l = 0.5$, as is consistent with the evidence in Hamilton (1994). Given these, we set the productivity levels z^h and z^l so as to match the consumption volatility of those agents who actually price the assets, as is suggested by the evidence on limited asset market participation (see, e.g., Parker (2001) and Attanasio and Vissing-Jorgensen (2003)). In our model bondholders' consumption is $c_t^e = u'^{-1}(1/z_t)$, so its volatility directly depends on the stochastic process for z_t. Following Ait-Sahalia et al. (2004), we

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[26]Recall from our analysis in Section 4.2 that the interaction term affects the short yield but not the yield on long (infinite-maturity) bonds, and consider how the interaction term in (22) is modified when agents hit by an idiosyncratic shock consume $\delta + \overline{W}$ instead of δ. For $\overline{W}$ small, we have:

$$\frac{(1-\alpha)u'(\delta + \overline{W})}{\alpha + (1-\alpha)u'(\delta + \overline{W})}\varepsilon^2 \simeq \underbrace{\frac{(1-\alpha)u'(\delta)}{\alpha + (1-\alpha)u'(\delta)}\varepsilon^2}_{\text{Interaction term (zero volume)}} + \underbrace{\frac{\alpha(1-\alpha)(-u''(\delta))}{(\alpha + (1-\alpha)u'(\delta))^2}\overline{W}\varepsilon^2}_{\text{Impact of volumes on interaction term}}$$

[27]The data we use are averages of the corresponding zero-coupon real yields over the period January 1997-October 2009. The monthly real zero-coupon yield curves are constructed by J. Huston McCulloch using US Treasury Inflation-Protected Securities (TIPS), and made available at `www.econ.ohio-state.edu/jhm/ts/ts.html#arch`.

proxy for the consumption volatility of relatively rich agents (the asset holders in the U.S) using that of luxury goods. The standard deviation of this variable is about 0.2 at annual frequency. Given our assumed utility function $u(.)$ and transition rates across aggregate states (π^l, π^h), this value is obtained for $z^h = 0.55591$ and $z^l = 0.4$.

We interpret δ as a replacement income for the unemployed and set it so that the replacement ratio be 40% of the average real wage. In our competitive economy with linear production function the average real wage is equal to the average value of z_t. Given the transition rates across aggregate states, this implies a mean real wage of about 0.5 and hence a replacement income of $\delta = 0.2$. We do not have data on the idiosyncratic risk faced by asset holders (again, a subset of the population), so we adopt an indirect calibration and choose (α, ρ) so as to match some features of the empirical real yield curve. Setting $\alpha = 0.9995$ and $\rho = 0.5$ allows our benchmark economy to match the average spread between the ten-year and one-year real interest rate in the U.S. ($= 0.92\%$).

We start with a zero net supply benchmark, and from then raise the supply of bonds. The second and third columns of Table 1 show the implied 1- and 10- year interest rates, respectively, while the fourth column shows the slope $r_{10} - r_1$. Under the calibration discussed above the shape of the yield curve in the benchmark, zero net supply economy (second row) exactly matches that of the empirical real yield curve. The third row of the Table shows how the shape of the yield curve changes as we raise the supply of bonds at all maturities from one year to ten years by 6.10^{-4}, which overall raises the level of the public debt to GDP ratio by one percentage point. The fourth row reports the implied changes in the level and slope of the curve –which in our example are raised by 3.6 and 0.1 basis points, respectively.[28]

Interest rates	r_1	r_{10}	$r_{10} - r_1$
Benchmark economy (%)	1.800	2.720	0.920
Economy with higher debt (%)	1.836	2.757	0.921
Change in interest rates (bp)	3.6	3.7	0.1

Table 1: Effect of a debt increase on the yield curve

In a separate technical appendix, we study a quantitative version of our model that relaxes some of its underlying assumptions. More specifically, we consider an economy in which i) the aggregate state is auto-regressive with continuous support, ii) agents do not instantaneously liquidate their asset wealth, iii) agents trade an additional asset with state-contingent income, iv) the tax structure is more general than in the baseline model, and v) public debt is contingent on the aggregate state.

The assumption of a continuous-support, autoregressive aggregate state (point i) completes our baseline analysis with discrete support and is required to solve the extended quantitative model via perturbation methods.[29] The property of a gradual asset liquidation (point ii) arises endoge-

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[28]Laubach (2009) documents that a one percentage point increase in the level of the U.S. debt to GDP ratio to raise interest rates by 3 to 4 basis points.

[29]As is further discussed in the technical appendix, the extended model does not enjoy the same tractability as the baseline theoretical model and must consequently be approximated around the steady state. We approximate the model at the third-order in order to preserve time-variations in risk premia.

nously when we assume that bond supplies are large, so that agents enjoy better self-insurance opportunities. In the technical appendix we focus on the case where the supply of bonds is such that in equilibrium it takes exactly two periods for the unemployed to hit the borrowing limit. The premium on long bonds in the economy with gradual liquidation responds to changes in the net supply of bonds exactly in the same way as in the baseline model.

In order to show that our theoretical results do not hinge on the limited set of assets that we consider (i.e., non-contingent debt), we evaluate their robustness by introducing an additional asset with state-contingent income (point iii), and by allowing for state-contingent bond supplies (point v). In so doing, we interpret the additional asset as a stock market index, so that dividends are procyclical, and consider a countercyclical fiscal policy whereby public debt rises and taxes fall during recessions (and the other way around during booms). We find that none of our theoretical results are altered by these assumptions, at least under the calibration that we consider. Finally, our baseline fiscal rule (equation (4)) assumes that only employed agents pay taxes. Having the unemployed pay part of the tax burden (point iv) not only relaxes our somewhat extreme baseline assumption, but also modifies the pricing kernel in a nontrivial way. More specifically, when purchasing bonds employed agents must take into account the additional risk that taxes may be high when they become unemployed (if bond prices are low, and consequently rolling over a constant stock of debt requires higher taxes). This *refinancing risk* feeds backs to all bond prices and tends to magnify the impact of the liquidation risk on the slope of the yield curve.

5 Concluding remarks

It is often claimed that the reason why long bonds command a premium over short bonds is that they are less "liquid", in the sense of being convertible into cash at more uncertain terms. In this paper, we have constructed a dynamic general-equilibrium model of the term structure under incomplete markets that is consistent with this view: holders of long bonds must be compensated for bearing the liquidation risk associated to these bonds, i.e., the risk that their price be low –due to unfavourable macroeconomic conditions– precisely when the holder needs to liquidate them –in order to buffer an adverse idiosyncratic income shock. Put differently, long, noncontingent bonds are a poor hedge against idiosyncratic shocks, because aggregate shocks cause the liquidation value of the bond portfolio –which explicitly enters bondholders' pricing kernel under incomplete markets and positive net bond supply– to covary with bond prices. In contrast, one-period bonds are a much better hedge, since they are not resold and, consequently, have a payoff that is completely independent of bondholders' wealth (of course, neither long nor short bonds are perfect hedges since both fails to be contingent on bondholders' individual state).

It seems natural, when considering the impact of liquidation risk on asset prices, to start by focusing on real, zero-coupon bonds, which by construction bear no income risk and only differ by their maturity. This strategy allowed us to isolate this specific source of risk and to distinguish it most clearly from the other sources of risk that potentially affect more complex financial

instruments. However, all long assets that are not indexed on agents' idiosyncratic state (most prominently: the stock market) potentially bear this risk, and for this reason should command a premium over short assets. Similarly, liquidation risk would be present in a monetary version of our framework that would generate a nominal yield curve –about which a wealth of evidence is available. We leave both of these lines of investigation for future research.

Acknowledgments

Appendix

A. Proof of proposition 1

1. Yield of infinite-maturity bonds. From (16) and the fact that z_t is bounded, we have $\lim_{k \to \infty} r_{t,k} = -\ln \beta \equiv r_\infty^{CM}$ (which is independent of the aggregate state).

2. Monotonicity of conditional yield curves. Let us denote $T^k = (T_{ij}^k)_{i,j=l,h}$, where T_{ij}^k is the ij's element of T^k (T to the k, with T given by (1)), that is, the probability of being in aggregate state j k periods ahead when the current state is i. Assumption A implies that the sequences $\{T_{hh}^k\}_{k=1}^\infty$ and $\{T_{ll}^k\}_{k=1}^\infty$, where $(T_{hh}^1, T_{ll}^1) = (\pi^h, \pi^l)$ and $(T_{hh}^{k+1}, T_{hh}^{k+1}) = (T_{hh}^k (\pi^h + \pi^l - 1) + 1 - \pi^l, T_{ll}^k (\pi^h + \pi^l - 1) + 1 - \pi^h)$, are positive and nondecreasing. From (16), the yield differences $r_\infty^{CM} - r_k^s$, $s = l, h$ are:

$$r_\infty^{CM} - r_k^h = \frac{1}{k} \ln \left[T_{hh}^k + \left(1 - T_{hh}^k \right) \frac{z^h}{z^l} \right] > 0, \ r_\infty^{CM} - r_k^l = \frac{1}{k} \ln \left[T_{ll}^k + \left(1 - T_{ll}^k \right) \frac{z^l}{z^h} \right] < 0. \quad \text{(A.1)}$$

Since $1/k$ is strictly decreasing in k while $T_{hh}^k + \left(1 - T_{hh}^k \right) \frac{z^h}{z^l}$ is nonincreasing in k (by the monotonicity of $\{T_{hh}^k\}_{k=1}^\infty$ and the fact that $z^h \geq z^l$), $r_\infty^{CM} - r_k^s$ is strictly decreasing in k. Hence, r_k^h lies below r_∞^{CM} and is strictly increasing in k. A symmetric argument applies to the yield curve in state l.

3. Monotonicity of the average yield curve. From (A.1), the average yield difference $\psi(k) \equiv \bar{r}_k - r^{CM}_\infty$ is given by:

$$\psi(k) = -\frac{1-\eta^l}{k} \ln\left(T^k_{hh} + \left(1 - T^k_{hh}\right)\frac{z^h}{z^l}\right) - \frac{\eta^l}{k} \ln\left(T^k_{ll} + \left(1 - T^k_{ll}\right)\frac{z^l}{z^h}\right), \quad \text{(A.2)}$$

with $\eta^l = \left(1 - \pi^h\right)/\left(2 - \pi^h - \pi^l\right)$. Diagonalising T, we may express $\left(T^k_{ll}, T^k_{hh}\right)$ as follows:

$$T^k_{hh} = \eta^l(\pi^h + \pi^l - 1)^k + 1 - \eta^l \quad \text{and} \quad T^k_{ll} = (1 - \eta^l)(\pi^h + \pi^l - 1)^k + \eta^l, \quad \text{(A.3)}$$

Substituting (A.3) into (A.2), we can write $\psi(k) = -\phi(k)/k$, where $\phi(k)$ is a function parameterised by π^l and π^h. Taking derivatives, we can show that $\partial(k^2\psi'(k))/\partial k = -k\phi''(k)$ is positive. This in turn implies that $\psi(k) < 0$ and $\psi'(k) > 0$, i.e., $\bar{r}_k$ lies below r^{CM}_∞ and is strictly increasing in k.

B. Proof of proposition 2

1. Pricing kernel From (12)–(13), we find that

$$p_{t,k} = \beta E_t\left[\left(\alpha_{t+1}\frac{z_t}{z_{t+1}} + (1 - \alpha_{t+1})z_t u'(\delta)\right)p_{t+1,k-1}\right], \quad k = 1, \ldots, n,$$

which provides the pricing factorization in m^{ZV}_{t+1} and I^{ZV}_{t+1}. From the literature on asset pricing with finite state-space (e.g., Mehra and Prescott (1985)), we conjecture (and verify) the existence of an equilibrium in which bond prices at any date t only depend on the current aggregate state s_t (and not on the whole history s^t, i.e., $p_{t,k}(s^t) = p_{t,k}$). With two aggregate states, bond prices are generated by the following recursions:

$$\frac{p^s_k}{z^s} = \beta\pi^s\left(\alpha^s + (1 - \alpha^s)z^s u'(\delta)\right)\frac{p^s_{k-1}}{z^s} + \beta(1 - \pi^s)\left(\alpha^{\bar{s}} + \left(1 - \alpha^{\bar{s}}\right)z^{\bar{s}}u'(\delta)\right)\frac{p^{\bar{s}}_{k-1}}{z^{\bar{s}}}, \quad s = l, h, \quad \text{(B.1)}$$

for $k = 1, \ldots, n$ and where $\bar{s}$ is the state opposite to s. From (13), at these prices unemployed agents face a binding borrowing constraint in state s if and only if:

$$p^s_k u'(\delta) > \beta\left(\pi^s\left(\frac{1 - \rho^s}{z^s} + \rho^s u'(\delta)\right)p^s_{k-1} + (1 - \pi^s)\left(\frac{1 - \rho^{\bar{s}}}{z^{\bar{s}}} + \rho^{\bar{s}}u'(\delta)\right)p^{\bar{s}}_{k-1}\right), \quad s = l, h.$$

Assumption C is a sufficient condition for these two inequalities to be satisfied when bond prices satisfy (B.1), so that unemployed agents do not participate in bond markets (i.e., they would like to *issue* bonds, but face a binding borrowing constraint, in either aggregate state).

2. Convergence towards a common limit. We first prove the following technical lemma.

Lemma 1. *Let $(u_n)_{n\geq 0}$, $(v_n)_{n\geq 0}$ be two real sequences such that $\begin{bmatrix} u_n & v_n \end{bmatrix}^\top = M\begin{bmatrix} u_{n-1} & v_{n-1} \end{bmatrix}^\top$, where M is a 2×2 real diagonalisable matrix whose eigenvalues λ_{max} and λ_{min} are such that*

$\lambda_{max} \geq \lambda_{min} > 0$. *Then,* $\left(-n^{-1}\ln(u_n)\right)_{n\geq 0}$ *and* $\left(-n^{-1}\ln(v_n)\right)_{n\geq 0}$ *converge towards the common limit* $\ln(\lambda_{max})$.

Proof: Diagonalising M, we may rewrite the recursion in the Lemma as

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = Q \begin{bmatrix} \lambda_{max}^n & 0 \\ 0 & \lambda_{min}^n \end{bmatrix} Q^{-1} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix},$$

where $(\lambda_{max}, \lambda_{min})$, $\lambda_{max} \geq \lambda_{min} \geq 0$, are the eigenvalues of M and Q the matrix of eigenvectors. $\frac{u_n}{\lambda_{max}^n}$ and $\frac{v_n}{\lambda_{max}^n}$ are affine functions of $\left(\frac{\lambda_{min}}{\lambda_{max}}\right)^n$, which is positive and either is equal to 1 or converges towards 0 as $n \to \infty$. Thus, $\frac{u_n}{\lambda_{max}^n}$ and $\frac{v_n}{\lambda_{max}^n}$ converge towards finite limits. We infer that $-\frac{1}{n}\ln(\frac{u_n}{\lambda_{max}^n})$ and $-\frac{1}{n}\ln(\frac{v_n}{\lambda_{max}^n})$ converge toward 0, and that $\lim_{n\to\infty} -\frac{1}{n}\ln(u_n) = \lim_{n\to\infty} -\frac{1}{n}\ln(v_n) = \ln(\lambda_{max})$∎

We may rewrite the bond price recursion in (B.1) in matrix form as follows:

$$[\ p_k^h/z^h \quad p_k^l/z^l\ ]^\top = M^{ZV}[\ p_{k-1}^h/z^h \quad p_{k-1}^l/z^l\ ]^\top \text{ for } k \geq 1, \text{ with } p_0^s = 1, \tag{B.2}$$

$$M^{ZV} = \beta \begin{pmatrix} \pi^h \kappa^h & \left(1-\pi^h\right)\kappa^l \\ \left(1-\pi^l\right)\kappa^h & \pi^l \kappa^l \end{pmatrix} \text{ and } \kappa^s \equiv \alpha^s + (1-\alpha^s)z^s u'(\delta). \tag{B.3}$$

Then, Lemma 1 implies that $\lim_{k\to\infty} r_k^s = r_\infty^{ZV}$, $s = h, l$, where

$$r_\infty^{ZV} = -\ln(\beta) - \ln(\nu_1), \text{ and} \tag{B.4}$$

$$\nu_{1/2} = \frac{1}{2}\left(\kappa^h \pi^h + \kappa^l \pi^l \pm \left((\kappa^h \pi^h + \kappa^l \pi^l)^2 - 4\kappa^h \kappa^l(\pi^h + \pi^l - 1)\right)^{\frac{1}{2}}\right). \tag{B.5}$$

3. Monotonicity of conditional yield curves. As stated in the proposition, a sufficient condition for the monotonicity of conditional yield curves in the incomplete-market, zero-volume case is that α^l and α^h be sufficiently close to each other. The necessary and sufficient condition is

$$\left(\pi^l + (\pi^h - 1)\frac{z^h}{z^l}\right)\left(\alpha^l + (1-\alpha^l)z^l u'(\delta)\right) \leq \left(\pi^h + (\pi^l - 1)\frac{z^l}{z^h}\right)\left(\alpha^h + (1-\alpha^h)z^h u'(\delta)\right), \tag{B.6}$$

and is indeed satisfied when α^l and α^h are close to each other, including when $\alpha^h = \alpha^l$. Under (B.6), the eigenvalues (ν_1, ν_2) of M^{ZV} in (B.5) satisfy

$$\nu_2 \leq \nu_1 \leq \pi^h \kappa^h + (1-\pi^h)\left(z^h/z^l\right)\kappa^l, \tag{B.7}$$

$$\nu_2 \leq \pi^h \kappa^h \leq \nu_1. \tag{B.8}$$

Since ν_1 is the largest eigenvalue, we have $\nu_2 \leq \nu_1$, while (B.6) implies that the second inequality in (B.7) also holds. The inequalities in (B.8) directly follow from (B.5) and the definition of κ^s in (B.3).

From (B.2), bond prices are given by $[\ p_k^h/z^h \quad p_k^l/z^l\ ]^\top = M^{ZV,k}[\ 1/z^h \quad 1/z^l\ ]^\top$, where $M^{ZV,k} \equiv$

$\left(M^{ZV}\right)^k$ (M^{ZV} to the k) is diagonalisable and can be written as

$$M^{ZV,k} = \beta^k P \begin{bmatrix} \nu_1^k & 0 \\ 0 & \nu_2^k \end{bmatrix} P^{-1}, \text{ with } P = \begin{bmatrix} (1-\pi^h)\kappa^l & (1-\pi^h)\kappa^l \\ \nu_1 - \pi^h\kappa^h & \nu_2 - \pi^h\kappa^h \end{bmatrix}.$$

From the latter recursion and equation (B.4), we find that $r_k^h - r_\infty^{ZV} = \tilde{\phi}(k)/k$, where

$$\tilde{\phi}(k) \equiv \ln\left[\nu_1 - \nu_2\right] - \ln\left[\frac{\nu_2^k}{\nu_1^k}\left(\nu_1 - \pi^h\kappa^h - (1-\pi^h)\frac{z^h}{z^l}\kappa^l\right) - \left(\nu_2 - \pi^h\kappa^h - (1-\pi^h)\frac{z^h}{z^l}\kappa^l\right)\right].$$

Condition (B.6) implies that $\frac{1}{k}\frac{\partial}{\partial k}\left(k^2\frac{\partial\left(r_k^h - r_\infty^{ZV}\right)}{\partial k}\right) > 0$ and successively that $k^2\frac{\partial\left(r_k^h - r_\infty^{ZV}\right)}{\partial k} > 0$ and $\frac{\partial\left(r_k^h - r_\infty^{ZV}\right)}{\partial k} > 0$: The yield curve is increasing in state h and converges from below to r_∞^{ZV}.

C. Proof of Proposition 3

Lemma 2. *Consider the following mean-preserving spread in aggregate and idiosyncratic risks around their unconditional means* (z,α): $\alpha^h = \alpha + 2\eta^l a$, $\alpha^l = \alpha - 2\left(1-\eta^l\right)a$, $z^h = z\left(1 + 2\eta^l\varepsilon\right)$ *and* $z^l = z\left(1 - 2\left(1-\eta^l\right)\varepsilon\right)$ *(where* $\eta^l = \left(1-\pi^l\right)/\left(2 - \pi^l - \pi^h\right)$*). A second-order development in* a *and* ε *gives*

$$\Delta^{ZV} = \frac{2(1-\pi^h)(1-\pi^l)(\pi^h+\pi^l)}{(2-\pi^h-\pi^l)^3(\alpha+(1-\alpha)zu'(\delta))^2}\left(-(3-\pi^h-\pi^l)(\pi^h+\pi^l-1)(zu'(\delta)-1)^2a^2\right.$$
$$\left. + 2(zu'(\delta)-1)(\alpha(2-\pi^h-\pi^l)^2 + (1-\alpha)zu'(\delta))\varepsilon\, a + \left(\alpha^2(2-\pi^h-\pi^l)^2 - (1-\alpha)^2z^2u'(\delta)^2\right)\varepsilon^2\right) \blacksquare$$

Proof. Let us first define $\Sigma\pi \equiv \pi^l + \pi^h$, $\Omega \equiv [(2-\Sigma\pi)(\alpha + (1-\alpha)zu'(\delta))]^{-1}$ and $\bar{s}$ the state opposite to s. From (20), the second-order expansion around $(a,\varepsilon) = 0^2$ of the conditional one-period yield gives

$$
\begin{aligned}
r_1^{ZV,s} = &-\ln(\beta) - \ln(\alpha + (1-\alpha)zu'(\delta)) \\
&\pm 2\Omega(1-\pi^s)\left[(\Sigma\pi-1)(zu'(\delta)-1)a - (\alpha(2-\Sigma\pi)+(1-\alpha)zu'(\delta))\varepsilon\right] \\
&+ 2\Omega^2(1-\pi^s)^2(\Sigma\pi-1)^2(zu'(\delta)-1)^2a^2 \\
&+ 4\Omega^2(1-\pi^s)\left[\pi^s(2-\Sigma\pi)^2(\alpha+(1-\alpha)zu'(\delta)) + (1-\pi^s)(\Sigma\pi-1)^2zu'(\delta)\right]a\varepsilon \\
&+ 2\Omega^2(1-\pi^s)\begin{bmatrix}(1-\pi^s)(\alpha(2-\Sigma\pi)+(1-\alpha)zu'(\delta))^2 \\ -2\alpha(1-\pi^{\bar{s}})(2-\Sigma\pi)(\alpha+(1-\alpha)zu'(\delta))\end{bmatrix}\varepsilon^2, & \text{(C.1)}
\end{aligned}
$$

where $\pm$ is a $-$ if $s = h$ and a $+$ if $s = l$. Moreover, the second-order expansion of r_∞^{ZV} in (B.4)

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gives:

$$
\begin{aligned}
r_\infty^{ZV} = & -\ln(\beta) - \ln(\alpha + (1-\alpha)zu'(\delta)) - 4\Omega^2(1-\pi^l)\eta^l \left[(\Sigma\pi - 1)(zu'(\delta) - 1)^2\right] a^2 \\
& + 4\Omega^2(1-\pi^l)\eta^l zu'(\delta) \left[(\Sigma\pi)(\alpha + (1-\alpha)zu'(\delta)) - 2(\Sigma\pi - 1)\right] a\varepsilon \\
& - 4\Omega^2(1-\pi^l)\eta^l \left[(\Sigma\pi - 1)\left((1-\alpha)zu'(\delta)\right)^2\right] \varepsilon^2.
\end{aligned}
\tag{C.2}
$$

The average short yield is $\bar{r}_1^{ZV} = \eta^l r_1^l + \left(1 - \eta^l\right) r_1^h$ and the slope is $\Delta^{ZV} = r_\infty^{ZV} - \bar{r}_1^{ZV}$.

1. Yield curve without aggregate shocks. Without aggregate shocks, (20) gives $r_k = -\ln(\beta) - \ln(\alpha + (1-\alpha)zu'(\delta))$, $k = 1, \ldots, n$.

2. Impact of α on level and the slope of the yield curve The level effect is proven by induction. From (B.2)–(B.3) and with $\alpha^l = \alpha^h = \alpha$, we have, for $k = 1$,

$$
\frac{\partial\left(p_1^s/z^s\right)}{\partial\alpha} = \beta\pi^s \left[\left(1 - z^s u'(\delta)\right)\frac{1}{z^s}\right] + \beta\left(1 - \pi^s\right)\left[\left(1 - z^{\bar{s}}u'(\delta)\right)\frac{1}{z^{\bar{s}}}\right], \quad s = l, h,
$$

which is negative since $1 - z^s u'(\delta) < 0$ (by Assumption B). For $k \geq 2$, we have

$$
\begin{aligned}
\frac{\partial\left(p_k^s/z^s\right)}{\partial\alpha} = & \ \beta\pi^s\left[\left(1 - z^s u'(\delta)\right)\frac{p_{k-1}^s}{z^s} + \left(\alpha + (1-\alpha)z^s u'(\delta)\right)\frac{\partial\left(p_{k-1}^s/z^s\right)}{\partial\alpha}\right] \\
& + \beta\left(1 - \pi^s\right)\left[\left(1 - z^{\bar{s}}u'(\delta)\right)\frac{p_{k-1}^{\bar{s}}}{z^{\bar{s}}} + \left(\alpha + (1-\alpha)z^{\bar{s}}u'(\delta)\right)\frac{\partial\left(p_{k-1}^{\bar{s}}/z^{\bar{s}}\right)}{\partial\alpha}\right],
\end{aligned}
$$

We know that $1 - z^s u'(\delta) < 0$, $p_{k-1}^s > 0$ and $\alpha + (1-\alpha)z^s u'(\delta) > 0$. Thus, if $\partial\left(p_{k-1}^s/z^s\right)/\partial\alpha < 0$ for $s = l$ and $s = h$, then $\partial\left(p_k^s/z^s\right)/\partial\alpha < 0$ for $s = l$ and $s = h$. Since $\partial\left(p_1^s/z^s\right)/\partial\alpha < 0$, $s = l, h$, this is true for all $k \geq 1$. It follows that a rise in idiosyncratic volatility (a fall in α) raises all bond prices, and thus lowers all bond yields, in both aggregate states. By implication, the same is true of the average yield curve.

From Lemma 2, when $a = 0$ we have

$$
\frac{\partial\Delta^{ZV}}{\partial\alpha} = 4\Omega^3(1-\pi^h)(1-\pi^l)(\Sigma\pi)\left[\alpha(2 - \Sigma\pi)^2 + (1-\alpha)zu'(\delta)\right]zu'(\delta)\varepsilon^2 > 0.
$$

3. Greater slope under incomplete markets/zero-volume than under complete markets
From Lemma 2, the slope is a quadratic concave function in a (negative coefficient in a^2) that admits a unique maximum. It is then easy to check that the maximal value of the slope is Δ_{max}^{ZV}, with:

$$
\Delta_{max}^{ZV} - \Delta^{CM} = \frac{2(2 - \Sigma\pi)(1-\pi^h)(1-\pi^l)(\Sigma\pi)}{(3 - \Sigma\pi)(\Sigma\pi - 1)}\varepsilon^2 > 0,
$$

where $\Delta^{CM} = 2\eta^l(1-\pi^l)(\Sigma\pi)\varepsilon^2$ is the slope in the complete-market case.

D. Proof of Proposition 4

1. Equilibrium portfolios. We proceed by construction: we first conjecture, and then derive a sufficient condition for, the existence of an equilibrium in which employed agents hold symmetric portfolios and never face a binding borrowing constraint, while unemployed are always borrowing-constrained and hold no bond. Formally, we conjecture, for $k = 1, ...n$:

$$e_t^i = 1 \Rightarrow \varphi_{t,k}^i = 0 \quad \text{and} \quad e_t^i = 0 \Rightarrow \varphi_{t,k}^i > 0. \tag{D.1}$$

Conjectured consumption levels and equilibrium pricing kernel. Consider first the consumption level of an unemployed agent at date t. If the agent was employed at date $t-1$, then from the budget constraint (7) and conjecture (D.1) the agent liquidates his entire portfolio and consume

$$c_t^i = \delta + \sum_{k=1}^n p_{t,k-1} b_{t-1,k}^i \ (> 0). \tag{D.2}$$

If, however, this agent was already unemployed at date $t-1$, then by (7) and (D.1) the agents consumes $c_t^{uu} = \delta > 0$.

Now consider the consumption level of an employed agent at date t. From (12), this agent enjoys marginal utility $1/z_t$ regardless of $e^{i,t}$ and consumes $c_t^e = u'^{-1}(1/z_t) \ (> 0)$. If this agent stays employed in the next period (which occurs with probability α), he will enjoy marginal utility $1/z_{t+1}$ (again by (12)). If, on the contrary, this agent moves into unemployment in the next period (which occurs with probability $1 - \alpha$), then his marginal utility will be $u'(c_{t+1}^i)$, where by construction c_{t+1}^i is given by (D.2). Then, substituting these marginal utilities into the intertemporal optimality conditions for employed agents (13) under conjecture (D.1), we obtain the following set of Euler equations:

$$\frac{p_{t,k}}{z_t} = \alpha\beta E_t \left[\frac{p_{t+1,k-1}}{z_{t+1}} \right] + (1-\alpha)\beta E_t \left[u'\left(\delta + \sum_{j=1}^n p_{t+1,j-1} b_{t,j}^i \right) p_{t+1,k-1} \right], \ k = 1, \ldots, n. \tag{D.3}$$

From (D.3), the bond demands $b_{t,j}^i$ are functions of aggregate variables only. Total supply being B_k, market clearing requires that $b_{t,k} = B_k/\omega^e$, meaning that no agent holds negative bond quantities. Substituting it in (D.3) together with (4), we express prices as a function of aggregate variables only.

Following the same steps, borrowing constraint condition (D.1) becomes:

$$p_{t,k} u'(\delta) > \beta(1-\rho) E_t \left[\frac{p_{t+1,k-1}}{z_{t+1}} \right] + \beta\rho E_t \left[p_{t+1,k-1} u'(\delta) \right], \tag{D.4}$$

where τ_t is given by (4). On the other hand, agents who were employed at date $t-1$ and who

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become unemployed at date t face a binding borrowing constraint if and only if, for all $k = 1, \ldots, n$:

$$p_{t,k} \, u' \left(\delta + \sum_{j=1}^{n} \frac{p_{t,j-1} B_j}{\omega^e} \right) > \beta(1 - \rho) E_t \left[\frac{p_{t+1,k-1}}{z_{t+1}} \right] + \beta \rho E_t \left[p_{t+1,k-1} u'(\delta) \right] \tag{D.5}$$

Since (D.5) implies (D.4), we only need to check that the equilibrium satisfies (D.5).

We prove the equilibrium existence in three steps. First, we derive initial conditions allowing us to avoid transitory dynamics. Second, we show the equilibrium exists when volumes are 0 and when there is no aggregate shocks. Third, we show by a continuity argument that the equilibrium exists when volumes and aggregate shocks are small. The technical part is the proof of continuity.

Conditions on agents' initial wealth. We assume that employed agents enter period 0 holding a quantity of bonds $b_{-1,k} = B_k/\omega^e$ with probability α, and no bond with probability $1 - \alpha$. Unemployed agents hold no bond with probability ρ, and $b_{-1,k} = B_k/\omega^e$ bonds with probability $1 - \rho$. The initial joint distribution of employment status and bond holdings is thus identical to the stationary distribution.

Existence of a no-trade equilibrium without aggregate shocks. If assets are in zero net supply, then there is no trade between agents and both the liquidation value of the portfolio and taxes will be equal to zero. Without aggregate uncertainty $z^h = z^l = z$, one easily finds the price of a one period bonds $p = m^{ZV}$, where m^{ZV} is given by (20). Rearranging (D.5) yields the following inequality: $(\alpha + (1 - \alpha)zu'(\delta)) zu'(\delta) > 1 - \rho + \rho \, zu'(\delta)$. Since $zu'(\delta) > 1$ by assumption B, the right hand side is maximum at $\rho = 1$, in which case the inequality remains true for any value of α; hence the no-trade equilibrium exists in the economy with zero volume and without aggregate risk.

Continuity of the yield curve w.r.t. bond supplies and aggregate shocks. Let us introduce the following change of variables, which greatly simplifies the algebra:

$$C_k^s = p_k^s / z^s, \quad s = h, l, \quad k = 1, \ldots, n. \tag{D.6}$$

Solving for C_k^s is equivalent to solving for prices (given the z^ss). We now define $B \equiv [B_n \ldots B_1]^\top$ as the vector of bond quantities for the n maturities, $Z \equiv \begin{bmatrix} z^l & z^h \end{bmatrix}^\top$ as the vector of productivity levels, and $C \equiv \begin{bmatrix} C_n^h & C_n^l \ldots C_0^h & C_0^l \end{bmatrix}^\top$ as the vector of price coefficients. 1_n and 0_n are vectors of length n containing respectively only ones and zeros. We then have the following lemma:

Lemma 3 (Equilibrium existence)**.** *There are neighbourhoods $\mathcal{B}$ of 0_n and $\mathcal{Z}$ of 1_2, such that if $B \in \mathcal{B}$ and $Z \in \mathcal{Z}$ then C is a $\mathcal{C}^1$ function of B and of Z.*

Proof. Let us first define $X \equiv [z^h \quad z^l \quad B^\top]$ and $v^s \equiv v \left(\delta + (z^s/\omega^e) \sum_{j=1}^{n} C_{j-1}^s B_j \right)$, whether $v = u'$ or u'' (for example, $u'^h \equiv u'(\delta + (z^h/\omega^e) \sum_{j=1}^{n} C_{j-1}^h B_j)$). Finally, let $1_{cond.}$ be the function

that takes value 1 when *cond.* is true and 0 otherwise, and

$$M(C,X) \equiv \beta \begin{bmatrix} \pi^h(\alpha + (1-\alpha)\, z^h\, u'^h) & (1-\pi^h)(\alpha + (1-\alpha)\, z^l\, u'^l) \\ (1-\pi^l)(\alpha + (1-\alpha)\, z^h\, u'^h) & \pi^l(\alpha + (1-\alpha)\, z^l\, u'^l) \end{bmatrix}. \tag{D.7}$$

Since $b_j^i = B_j/\omega^e$, (D.3) can be written as follows:

$$[\, C_k^h \quad C_k^l \,]^\top = M(C,X) \cdot [\, C_{k-1}^h \quad C_{k-1}^l \,]^\top \text{ for } k = 1, \dots, n. \tag{D.8}$$

By stacking equalities, we rewrite (D.8) as $f(C,X) = 0_{(2n+2)\times 1}$, where f is the following $\mathcal{C}^1$ function:

$$f(C,X) \equiv C - \begin{bmatrix} 0_{2\times 2} & M(C,X) & 0_{2\times 2} & \dots & 0_{2\times 2} \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & M(C,X) \\ 0_{2\times 2} & & & \dots & 0_{2\times 2} \end{bmatrix} C - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1/z^h \\ 1/z^l \end{bmatrix}.$$

To prove that C is a $\mathcal{C}^1$ function of B and Z, we show that the Jacobian of f w.r.t. C is invertible. The derivatives of f w.r.t. $(C_{n-i}^s)_{i=0,\dots,n}$ can be written as $\partial f/\partial C_{n-i}^s = \Gamma_{n-i}^s + K_{n-i}^s$, $i = 0, \dots, n$, with Γ_{n-i}^s defined, for $i = 0$, as $\Gamma_n^s \equiv [1_{s=h}, \; 1_{s=l}, 0_{2n}]^\top$ and, for $i = 1, \dots, n$, as

$$\Gamma_{n-i}^s \equiv \begin{bmatrix} 0_{2(i-1)\times 1} \\ -\beta\,(\alpha + (1-\alpha)z^s u'^s)\,\widetilde{\pi}^{hs} \\ -\beta\,(\alpha + (1-\alpha)z^s u'^s)\,\widetilde{\pi}^{ls} \\ 1_{s=h} \\ 1_{s=l} \\ 0_{2(n-i)\times 1} \end{bmatrix} \begin{array}{l} \\ \\ \\ \longleftarrow \text{ Rank } 2i+1 \\ \longleftarrow \text{ Rank } 2i+2, \\ \\ \end{array}$$

where $\widetilde{\pi}^{hh} \equiv \pi^h, \widetilde{\pi}^{lh} \equiv 1 - \pi^l, \widetilde{\pi}^{ll} \equiv \pi^l, \widetilde{\pi}^{hl} \equiv 1 - \pi^h$ and

$$K_{n-i}^s \equiv -\beta\,(1-\alpha)\,[B_{n+1-i}\,1_{i>0}/\omega^e]\,(z^s)^2\,u''^s$$
$$\times \left[\widetilde{\pi}^{hs}\,C_{n-1}^s,\; \widetilde{\pi}^{ls}\,C_{n-1}^s \dots, \widetilde{\pi}^{hs}\,C_0^s,\; \widetilde{\pi}^{ls}\,C_0^s,\; 0,\; 0\right]^\top \text{ for } i = 0, \dots, n.$$

The Jacobian $Df_Y = (\frac{\partial f}{\partial C_n^h}, \frac{\partial f}{\partial C_n^l}, \dots, \frac{\partial f}{\partial C_{n-i}^h}, \frac{\partial f}{\partial C_{n-i}^l}, \dots, \frac{\partial f}{\partial C_0^h}, \frac{\partial f}{\partial C_0^l})$ of f w.r.t. to C can be expressed as the sum of an upper triangular matrix with only 1s on its diagonal and a matrix that is equal to 0 when $B = 0$ (because $K_{n-i}^s = 0$ if $B = 0$). The Jacobian is thus invertible for $B = 0$. Then, the implicit function theorem allows us to prove that C is a continuous (in fact $\mathcal{C}^1$) function of $[B^\top \; Z^\top]$ in a neighbourhood V_1 of $[0_n^\top \; 1_2^\top]$. Moreover, if $[B^\top \; Z^\top] = [0_n^\top \; 1_2^\top]$, then C satisfies conditions (D.5). By continuity, there exists a neighbourhood $V_2 \subset V_1$, such that condition (D.5) is satisfied if $[B^\top \; Z^\top] \in V_2$. ∎

The lemma establishes that, starting from a no uncertainty/zero net supply situation, a gradual

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increase in aggregate risk or bond supplies does not cause the yield curve to jump. Since the equilibrium exists in the zero-volume/no aggregate uncertainty case, it also exists when volumes and aggregate risk are sufficiently small (that is, (D.5) holds).

2. Pricing kernel decomposition   Substituting the market-clearing condition $b_{t,k} = B_k/\omega^e$ into the Euler equation (D.3) and rearranging gives the bond-pricing equation and the corresponding pricing kernel components in the proposition.

E. Proof of Proposition 5

1a. Impact of bond supplies on the level of the yield curve.   We prove the result by induction. Taking the derivative of (D.8) w.r.t. to B_i, $1 \leq i \leq n$, we get:

$$\frac{\partial C_k^\zeta}{\partial B_i} = \beta \sum_{s=h,l} \widetilde{\pi}^{\zeta s} \left[(\alpha + (1-\alpha)z^s u'^s) \frac{\partial C_{k-1}^s}{\partial B_i} + \frac{(1-\alpha)C_{k-1}^s(z^s)^2 u''^s}{\omega^e} \sum_{j=1}^n \left(\frac{\partial C_{j-1}^s}{\partial B_i} B_j + C_{i-1}^s \right) \right]$$
(E.1)

where u'^s and u''^s are defined in the proof of Lemma 3, and the $\widetilde{\pi}$s are as in Lemma 3.

- The result holds for $k = 1$, since for small bond supplies we have

$$\frac{\partial C_1^\zeta}{\partial B_i} \approx \beta \frac{(1-\alpha)\,u''\,(\delta)}{\omega^e} \sum_{s=h,l} \widetilde{\pi}^{\zeta s} z^s C_{i-1}^s < 0.$$

- Suppose that the result holds for $k-1$: $\frac{\partial C_{k-1}^h}{\partial B_i}, \frac{\partial C_{k-1}^l}{\partial B_i} < 0$. Since C_{j-1}^s is a $\mathcal{C}^1$ function of B_i, $\frac{\partial C_{j-1}^s}{\partial B_i}$ is continuous in B_i and $B_j \frac{\partial C_{j-1}^s}{\partial B_i}$ is negligible relative to C_{i-1}^s for small bond supplies. Then, (E.1) implies $\frac{\partial C_k^h}{\partial B_i} < 0$, so that greater bond supply decreases prices (i.e., raises yields).

1b. Impact of bond supplies on the slope of the yield curve.   Diagonalising $M(C, X)$ in (D.7), we get $M(C, X) = \beta Q\, D Q^{-1}$, where Q is a invertible matrix and $D = Diag(\tilde{\nu}_1, \tilde{\nu}_2)$, with

$$\tilde{\nu}_1 = H + \tilde{\nu}_2 = \frac{1}{2}\left(\alpha\left(\pi^h + \pi^l\right) + (1-\alpha)\left(z^h u'^h \pi^h + z^l u'^l \pi^l\right) + H \right), \text{ and}$$

$$H \equiv \left[\begin{array}{c} \left(\alpha(\pi^h + \pi^l) + (1-\alpha)(z^h \pi^h\, u'^h + z^l \pi^l\, u'^l)\right)^2 \\ -4(\pi^h + \pi^l - 1)(\alpha + (1-\alpha)z^h\, u'^h)(\alpha + (1-\alpha)z^l\, u'^l) \end{array} \right]^{1/2} > 0.$$

Using Lemma 1, the long yield r_∞^{PV} is given by:

$$\lim_{k \to \infty} r_k^h = \lim_{k \to \infty} r_k^l = -\ln\beta - \ln(\tilde{\nu}_1).$$
(E.2)

From (D.7)–(D.8) and the fact that $C_0^s = 1/z^s$, the short yield in state s is:

$$r_1^s = -\ln p_1^s = -\ln\beta - \ln\left[\pi^s(\alpha + (1-\alpha)z^s u'^s) + (1-\pi^s)(\alpha z^s/z^{\bar{s}} + (1-\alpha)z^s u'^{\bar{s}})\right], \ s = l, h,$$

where $\bar{s}$ is the state opposite to s.

As in proof of Proposition 3, we consider a mean preserving spread in z and carry out a second-order Taylor expansion of the derivative of the slope of the yield curve w.r.t. to an increase in bond supplies of maturity j (i.e., $\frac{\partial\Delta^{PV}}{\partial B_j} = \frac{\partial\left(r_\infty^{PV}-\left((1-\eta^l)r_1^h+\eta^l r_1^l\right)\right)}{\partial B_j}$) around $\varepsilon = 0$ and zero net volumes. The next section presents a second-order Taylor expansion of Δ^{PV} for the case of i.i.d. shocks, which leads to the expressions in (26)–(27). For a general shock process, we only focus on the expansion of $\frac{\partial\Delta^{PV}}{\partial B_j}$. We just state the result here and leave the proof in the separate technical appendix. In the general case, the derivative of the slope of the yield curve w.r.t. to an increase in bond supplies of maturity j is found to be:

$$
\left.\frac{\partial\Delta^{PV}}{\partial B_j}\right|_{B_k=0,\ k=1,\dots,n} = \frac{4\Omega^2(1-\pi^h)(1-\pi^l)(\Sigma\pi)\left((2-\Sigma\pi)^2\alpha+(1-\alpha)zu'(\delta)\right)}{(2-\Sigma\pi)(\alpha+(1-\alpha)zu'(\delta))^{2-j}}
$$
$$
\times \frac{(1-\alpha)\left(-zu''(\delta)\right)}{\omega^e}\beta^{j-1}\left(\alpha+1_{j>1}\sum_{i=0}^{j-2}(\Sigma\pi-1)^i\left((2-\Sigma\pi)\alpha+(1-\alpha)zu'(\delta)\right)\right)\varepsilon^2,
$$

with $1_{j>1} = 1$ if $j > 1$ and 0 if $j = 1$, and where $\Sigma\pi = \pi^l+\pi^h$ and $\Omega \equiv [(2-\Sigma\pi)(\alpha+(1-\alpha)zu'(\delta))]^{-1}$. The impact on volumes the slope is thus positive.

2. Impact on the slope of an increase in the relative supply of short bonds. From the latter expression we have $\partial\Delta^{PV}/\partial B_1 < \partial\Delta^{PV}/\partial B_{j>1}$, an increase in the supply of one-period bonds increases less the slope of the yield curve than an increase in the supply of bonds of any maturity greater than one.

F. Positive net bond supplies: explicit formulas in the i.i.d. case

First, we have $C_0^h = (1+\varepsilon)^{-1}$ and $C_0^l = (1-\varepsilon)^{-1}$. Using the recursion (D.7)–(D.8) to compute C_1^s, $s = l, h$ and rearranging, we find

$$
C_1^s = \frac{\alpha\beta}{1-\varepsilon^2} + (1-\alpha)\beta u'(\delta) + \frac{(1-\alpha)\beta u''(\delta)}{2}\left[(1+\varepsilon)\sum_{j=1}^{n}C_{j-1}^h\frac{B_j}{\omega^e} + (1-\varepsilon)\sum_{j=1}^{n}C_{j-1}^l\frac{B_j}{\omega^e})\right],
$$

which in turn implies that $C_1^h = C_1^l \equiv C_1$. The same recursion gives, for $j \geq 2$,

$$
\frac{C_j^s}{C_{j-1}^s} = \alpha\beta + \beta(1-\alpha)u'(\delta) + \beta\frac{(1-\alpha)u''(\delta)}{2}\left[(1+\varepsilon)\sum_{j=1}^{n}C_{j-1}^h\frac{B_j}{\omega^e} + (1-\varepsilon)\sum_{j=1}^{n}C_{j-1}^l\frac{B_j}{\omega^e})\right].
$$

By induction, $C_j^h = C_j^l \equiv C_j$ for all $j \geq 1$, so the latter two equations can be written as:

$$C_1 = \frac{\alpha\beta}{1-\varepsilon^2} + (1-\alpha)\beta u'(\delta) + (1-\alpha)\beta u''(\delta)\left(\frac{B_1}{\omega^e} + \sum_{j=2}^{n} C_{j-1}\frac{B_j}{\omega^e}\right), \quad \text{(F.1)}$$

$$\frac{C_j}{C_{j-1}} = \beta\alpha + \beta(1-\alpha)u'(\delta) + \beta(1-\alpha)u''(\delta)\left(\frac{B_1}{\omega^e} + (1+\varepsilon^2)\sum_{j=2}^{n} C_{j-1}\frac{B_j}{\omega^e}\right). \quad \text{(F.2)}$$

Equations (F.1)–(F.2) define a system of n equations with n unknown, the C_js. The solution to this system expresses the vector $[C_j]_{j=1}^n$ as a function of ε^2, and for small shocks we have $C_j \simeq \overline{C}_j + \left(\partial_\varepsilon^2 C_j\right)\varepsilon^2$, $j = 1, \ldots, n$, where $\overline{C}_j$ is the value of C_j without aggregate shocks and $\left(\partial_\varepsilon^2 C_j\right) \equiv \partial C_j/\partial\varepsilon^2\big|_{\varepsilon^2=0}$ (both the $\overline{C}_j$s and the $\left(\partial_\varepsilon^2 C_j\right)$s are undetermined coefficients at this stage). Moreover, we define $\overline{W} \equiv \frac{1}{\omega^e}\sum_{j=1}^n \bar{p}_{j-1}B_j = \frac{1}{\omega^e}\sum_{j=1}^n \overline{C}_{j-1}B_j$ as the value of the portfolio without aggregate shocks, $\overline{W}_2 \equiv \overline{W} - B_1/\omega^e = \frac{1}{\omega^e}\sum_{j=2}^n \overline{C}_{j-1}B_j$ the same value excluding holdings of one-period bonds, and $\left(\partial_\varepsilon^2 W\right) \equiv \frac{1}{\omega^e}\sum_{j=2}^n \left(\partial_\varepsilon^2 C_{j-1}\right)B_j$ as the change in the value of the portfolio following a marginal change in ε^2. Computing the first-order approximations to the right hand sides of (F.1)–(F.2) around $\varepsilon^2 = 0$, we get

$$C_1 \simeq \underbrace{\alpha\beta + (1-\alpha)\beta u'(\delta) + (1-\alpha)\beta u''(\delta)\overline{W}}_{=\bar{C}_1} + \underbrace{\left(\alpha\beta + (1-\alpha)\beta u''(\delta)\left(\partial_\varepsilon^2 W\right)\right)}_{=\left(\partial_\varepsilon^2 C_1\right)}\varepsilon^2, \quad \text{(F.3)}$$

$$\frac{C_j}{C_{j-1}} = \underbrace{\alpha\beta + (1-\alpha)\beta u'(\delta) + (1-\alpha)\beta u''(\delta)\overline{W}}_{=\bar{C}_1} + \underbrace{\beta(1-\alpha)u''(\delta)\left(\overline{W}_2 + \left(\partial_\varepsilon^2 W\right)\right)}_{\equiv\mu}\varepsilon^2. \quad \text{(F.4)}$$

From (F.4), we have, for $j \geq 2$, $C_j = C_{j-1}\bar{C}_1 + C_{j-1}\mu\varepsilon^2$. Using this recursion starting at $C_1 = \bar{C}_1 + \left(\partial_\varepsilon^2 C_1\right)\varepsilon^2$ and neglecting terms in ε^4, we find that, for $j \geq 1$,

$$C_j \simeq \left(\overline{C}_1\right)^j + \left(\overline{C}_1\right)^{j-1}\left(\left(\partial_\varepsilon^2 C_1\right) + (j-1)\mu\right)\varepsilon^2,$$

where $\bar{C}_j = \left(\overline{C}_1\right)^j$. Now substitute the values for $\left(\partial_\varepsilon^2 C_1\right)$ and μ in (F.3) and (F.4) into the latter expression to find

$$C_j \simeq \left(\overline{C}_1\right)^j + \left(\overline{C}_1\right)^{j-1}\left(\alpha\beta + (1-\alpha)\beta u''(\delta)\left[(j-1)\overline{W}_2 + j\left(\partial_\varepsilon^2 W\right)\right]\right)\varepsilon^2.$$

For small bond volumes, the terms in $\overline{W}$, $\overline{W}_2$ and $\left(\partial_\varepsilon^2 W\right)$ (which include the B_js) are second-order relative to $\alpha\beta$, so the latter equation gives $C_j \simeq \left(\overline{C}_1\right)^j + \alpha\beta\left(\overline{C}_1\right)^{j-1}\varepsilon^2$, where $\overline{C}_1 \simeq \alpha\beta + (1-\alpha)\beta u'(\delta)$ (by (F.3)). Since $C_j \simeq \overline{C}_j + \left(\partial_\varepsilon^2 C_j\right)\varepsilon^2$, this implies that $\left(\partial_\varepsilon^2 C_j\right) \simeq \alpha\beta\left(\overline{C}_1\right)^{j-1}$, which in turn gives $C_j \simeq \left(\overline{C}_1\right)^j + \alpha\beta\left(\overline{C}_1\right)^{j-1}\varepsilon^2$. We infer $\left(\partial_\varepsilon^2 W\right)$ to be:

$$\left(\partial_\varepsilon^2 W\right) = \sum_{j=2}^{n}\frac{\left(\partial_\varepsilon C_{j-1}\right)B_j}{\omega^e} \simeq \sum_{j=2}^{n}\frac{\alpha\beta\left(\overline{C}_1\right)^{j-2}B_j}{\omega^e} = \frac{\alpha\beta\sum_{j=2}^n\left(\overline{C}_1\right)^{j-1}B_j}{\omega^e\overline{C}_1} = \frac{\alpha\overline{W}_2}{\alpha + (1-\alpha)u'(\delta)}. \quad \text{(F.5)}$$

From (E.2), the long yield in the i.i.d. case is

$$r_\infty^{PV} = -\ln(\beta) - \ln\left(\alpha + \frac{1-\alpha}{2}((1+\varepsilon)u'^h + (1-\varepsilon)u'^l)\right), \tag{F.6}$$

with $u'^s = u'\left(\delta + \frac{1}{\omega^e}\sum_{j=1}^n p_{j-1}^s B_j\right)$. Since $p_{j-1}^s = C_{j-1}z^s$ for $j \geq 2$ and $p_0^s = 1$, we have

$$u'^s = u'\left(\delta + \frac{B_1}{\omega^e} + \frac{1\pm\varepsilon}{\omega^e}\sum_{j=2}^n C_{j-1}B_j\right) \simeq u'(\delta) + u''(\delta)\left(\frac{B_1}{\omega^e} + \frac{1\pm\varepsilon}{\omega^e}\sum_{j=2}^n C_{j-1}B_j\right),$$

and hence, again neglecting terms in ε^4,

$$\frac{(1+\varepsilon)u'^h + (1-\varepsilon)u'^l}{2} = u'(\delta) + u''(\delta)\left(\frac{B_1}{\omega^e} + \frac{1+\varepsilon^2}{\omega^e}\sum_{j=2}^n C_{j-1}B_j\right)$$

$$\simeq u'(\delta) + u''(\delta)\left(\bar{W} + \frac{1}{\omega^e}\cdot\frac{\partial(1+\varepsilon^2)\sum_{j=2}^n C_{j-1}B_j}{\partial\varepsilon^2}\cdot\varepsilon^2\right) \simeq u'(\delta) + u''(\delta)\left(\bar{W} + \left(\bar{W}_2 + (\partial_\varepsilon^2 W)\right)\varepsilon^2\right).$$

Substituting this expression into (F.6) and using the value of $(\partial_\varepsilon^2 W)$ in (F.5), we find

$$r_\infty^{PV} = -\ln(\beta) - \ln\left(\alpha + (1-\alpha)u'(\delta) + (1-\alpha)u''(\delta)\left(\bar{W} + \bar{W}_2\varepsilon^2 + \frac{\alpha\bar{W}_2}{\alpha + (1-\alpha)u'(\delta)}\varepsilon^2\right)\right) \tag{F.7}$$

The linearisation of (F.7) around $(\bar{W}, \bar{W}_2) = (0,0)$, with $\bar{W}_2 = \bar{W} - B_1/\omega^e = \bar{W} - b_1$, gives (26) in the body of the paper.

Let us now turn to the short yield. Under i.i.d. shocks, the average short yield is $\bar{r}_1^{PV} = -\frac{1}{2}\sum_{s=l,h}\ln C_1 z^s = -\ln C_1 - \frac{\ln(1-\varepsilon^2)}{2}$. With ε^2 small, we have $-\ln(1-\varepsilon^2)/2 \simeq \varepsilon^2/2$, while C_1 is given by (F.3) and $(\partial_\varepsilon^2 W)$ by (F.5)). This gives:

$$\bar{r}_1^{PV} \simeq \frac{\varepsilon^2}{2} - \ln\beta - \ln\left(\alpha + (1-\alpha)u'(\delta) + (1-\alpha)u''(\delta)\bar{W} + \alpha\varepsilon^2 + \frac{\alpha(1-\alpha)u''(\delta)\bar{W}_2}{\alpha + (1-\alpha)u'(\delta)}\varepsilon^2\right)$$

Linearising the latter expression around $(\bar{W}, \bar{W}_2) = 0^2$, we obtain

$$\begin{aligned}
r_1^{PV} \simeq {}& -\ln(\beta) + \frac{\varepsilon^2}{2} - \ln\left(\alpha + (1-\alpha)u'(\delta) + \alpha\varepsilon^2\right) \\
& -\frac{(1-\alpha)u''(\delta)\bar{W}}{\alpha + (1-\alpha)u'(\delta) + \alpha\varepsilon^2} - \frac{\alpha(1-\alpha)u''(\delta)\varepsilon^2}{[\alpha + (1-\alpha)u'(\delta) + \alpha\varepsilon^2][\alpha + (1-\alpha)u'(\delta)]}\bar{W}_2. \tag{F.8}
\end{aligned}$$

For small ε^2 small, this expression gives (27) in the body of the paper.

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